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Alias-Free Wigner Distribution Function and Complex Ambiguity Function for Discrete-Time Samples

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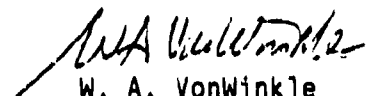Newport, Rhode Island / New London, Connecticut

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Preface

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If an arbitrary complex continuous waveform s(t) with finite overall frequency extent F Hertz is sampled with time increment $\Delta < 1/F$, the aliasing can be controlled and the continuous time waveform s(t) reconstructed exactly at any desired time instant from waveform samples $\{s(k\Delta)\}$. On the other hand, it is commonly believed that aliasing of the corresponding Wigner distribution function (WDF) can only be avoided by sampling twice as fast; i.e., $\Delta < (2F)^{-1}$ is thought to be required.

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Interspersed Aliasing Lobes
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Spectral Correlation
Ambiguity Function

19. (Cont'd.)

— Alternatively, interpolation of the time data has been suggested as a means of circumventing aliasing of the WDF; however, the computational burden has proven excessive if done by sinc function interpolation.

It is demonstrated here that this conjecture is false, and that the usual sampling criterion, $\Delta < 1/F$, suffices for exact reconstruction of the original continuous WDF, as well as the complex ambiguity function (CAF), at all time, frequency locations, without an excessive amount of computational effort. The inadequacy of earlier investigations was due to incomplete processing of all the information available in data samples $\{s(k\Delta)\}$. Correct processing eliminates the troublesome close-in aliasing lobes, leaving only the standard aliasing lobes that can be suppressed if sampling increment $\Delta < 1/F$. The new feature is a diamond-shaped gating function in the two-frequency domain, where interspersed aliasing lobes occur.

The required data processing for an alias-free WDF and CAF is strikingly simple. It requires that the available time data be immediately transformed to the frequency domain, and that the frequency domain versions of the WDF and CAF integrals be employed, rather than the time domain forms. Discretization of the reconstructed alias-free WDF and CAF in both time and frequency is then investigated and the required FFT sizes and ranges of variables are determined. Interpolation of samples $\{s(k\Delta)\}$, or reconstruction of s(t) from these samples, is not necessary nor utilized.

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

t	time
$s(t)$	complex waveform
f	frequency
$S(f)$	voltage-density spectrum of $s(t)$, (13)
F	total bandwidth of $S(f)$, (14) and figure 1
Δ	time sampling increment, (15)
$\vec{S}(f)$	approximation to $S(f)$, (16)
$W(t,f)$	WDF, (26), (33)
τ	time separation, (26), (32)
v	frequency separation, (26), (35)
$R(t,\tau)$	TCF, (32)
$\chi(v,\tau)$	CAF, (34)
$A(v,f)$	SCF, (35)
$W_a(t,f)$	approximate WDF, (40) and (41)
$A_a(v,f)$	approximate SCF, (46) and (58)
$D(v,f)$	diamond gating function, (53)
$\chi_a(v,\tau)$	approximate CAF, (56) and (57)
$\bar{S}(f)$	spectrum computed from time samples, (69)
T	overall effective duration of $s(t)$, (84) and figure 9
N	FFT size, (86)
$\tilde{W}(t,f)$	frequency domain WDF approximation, (91)
M	size of FFT, (99), (113)
$\hat{s}(t)$	time aliased waveform, (114) and figure 11
$\hat{W}(t,f)$	time domain WDF approximation, (123)

LIST OF SYMBOLS (Cont'd)

sub i	impulsive approximations, (B-1),(B-2)
$d(t,f)$	interpolation function in time t, (C-3)
$\mathcal{d}(v,\tau)$	interpolation function in delay τ, (C-6)
$\mathcal{g}(t,\tau)$	two-dimensional interpolation function, (C-9)

ABBREVIATIONS

FFT	Fast Fourier Transform, (12)
WDF	Wigner Distribution Function, (26),(33)
TCF	Temporal Correlation Function, (32)
CAF	Complex Ambiguity Function, (34)
SCF	Spectral Correlation Function, (35)

ALIAS-FREE WIGNER DISTRIBUTION FUNCTION AND
COMPLEX AMBIGUITY FUNCTION FOR DISCRETE-TIME SAMPLES

INTRODUCTION

The attributes of the Wigner distribution function (WDF) have come under close scrutiny in recent years; see, for example, [1,2,3] and the references listed therein. However, the numerical calculation of the WDF from discrete time data still suffers from the belief that the sampling rate of a given time waveform must be twice as large for computation of an alias-free WDF, as the rate required for reconstruction of the original continuous waveform. If true, this would double the number of data points that must be collected to cover a given time interval, and greatly increase the number of subsequent computations. This contention applies to the complex analytic waveform as well as to a real waveform.

It is the purpose of this report to establish the fact that the sampling rate need not be doubled, and that an alias-free WDF, as well as complex ambiguity function (CAF), can still be quickly and efficiently obtained, provided that all the information in the available data stream is extracted and properly processed. Some recent effort on this topic [4,5,6] did not discover the particular complete set of processing required, leading to the conjecture [5, page 1068] that it was not possible to accomplish the desired goal for the WDF.

We will show not only that the desired goal can be achieved, but that the required data processing for an alias-free WDF and CAF is strikingly simple. Our approach to the solution initially involves the four time/frequency domains associated with the WDF and its various Fourier transforms. However, in hindsight, an extremely simple and direct method of obtaining the WDF and CAF will be presented, which requires only FFTs (fast Fourier transforms) for its implementation.

An alias-free discrete WDF and CAF have been achieved in [7] and [8], by means of interpolating either the waveform time samples or the spectrum frequency samples. Also, the ranges and required increment sizes in time and frequency of the various two-dimensional functions have been carefully scrutinized in [7], by discrete Fourier transform techniques. However, that approach does not illuminate how the various aliasing lobes interact and can be controlled. Furthermore, we utilize a continuous approximation approach (rather than a discrete Fourier technique), which lends tremendous insight into the shortcomings of current processing methods and brings out the fundamental properties of the various two-dimensional functions and their domains of definition. The final discretization in time and frequency is only done with deference to practical computer evaluation.

In fact, we will not define a discrete WDF or CAF here. Instead, we attempt to recover the WDF and CAF of the original continuous time waveform, by developing approximations and then controlling or eliminating the errors in these approximations. Only after this is accomplished, do we then address discretization of the time and frequency arguments of the two-dimensional functions of interest.

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NOTATION

For economy of presentation, a number of notational and manipulative shortcuts are employed here. We have collected them all together at this point, and will employ them freely later, with minimal comment. We define

$$
rect(x) = \begin{cases} 1 & \text{for } |x| < 1/2 \\ 0 & \text{otherwise} \end{cases} , \tag{1}
$$

$$
sinc(x) = \frac{\sin(\pi x)}{\pi x} \quad \text{for all } x . \tag{2}
$$

The symbols $\int$ and $\sum$ without limits denote that integration and summation are to be conducted over the complete range of nonzero integrand and summand, respectively.

The convolution of two functions $g(x)$ and $h(x)$ is denoted by

$$
g(x) \circledast h(x) = \int du \; g(u) \; h(x - u) . \tag{3}
$$

The two-dimensional convolution of two functions is

$$
g(x,y) \overset{xy}{\circledast} h(x,y) = \iint du \; dv \; g(u,v) \; h(x - u, y - v) . \tag{4}
$$

The Fourier transform of a time domain function $s(t)$ into its spectrum in the frequency domain f is according to the pair of relations

$$
S(f) = \int dt \; \exp(-i2\pi ft) \; s(t) ,
$$

$$
s(t) = \int df \; \exp(i2\pi ft) \; S(f) . \tag{5}
$$

3

Then the Fourier transform of a product of time functions is equal to the convolution of the two spectra:

$$\int dt \; \exp(-i2\pi ft) \; a(t) \; b(t) \; = \; A(f) \circledast B(f) \; . \tag{6}$$

The infinite impulse train in time t, with spacing Δ, is

$$\Delta \sum_n \delta(t - n\Delta) \; , \quad n \text{ integer} \; . \tag{7}$$

Its Fourier transform is another infinite impulse train in frequency f, with reciprocal spacing:

$$\int dt \; \exp(-i2\pi ft) \; \Delta \sum_n \delta(t - n\Delta) \; = \; \sum_n \delta(f - \frac{n}{\Delta}) \; . \tag{8}$$

Combination of (6) and (8) leads to a very useful relation that is employed frequently in the following:

$$\int dt \; \exp(-i2\pi ft) \; a(t) \; \Delta \sum_n \delta(t - n\Delta) \; =$$

$$= \; A(f) \circledast \sum_n \delta(f - \frac{n}{\Delta}) \; = \; \sum_n A(f - \frac{n}{\Delta}) \; . \tag{9}$$

The discrete Fourier transform operation arises frequently; consider

$$Z(n) \equiv \sum_k \exp(-i2\pi kn/N) \; z(k) \quad \text{for all } n \; . \tag{10}$$

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The periodicity of $Z(n)$ means that it only need be computed for one period, namely $0 \leq n \leq N - 1$. The absence of limits on the sum in (10) means that it goes from $k = -\infty$ to $+\infty$. However, since $z(k)$, $z(k \pm N)$, $z(k \pm 2N)$, ... all receive the same weight in (10), regardless of the value of n, the values of $\{z(k)\}$ can be "collapsed" according to

$$\tilde{z}(k) \equiv \left\{ \begin{array}{ll} \sum_j z(k + jN) & \text{for } 0 \leq k \leq N - 1 \\ \\ 0 & \text{otherwise} \end{array} \right\} . \tag{11}$$

and (10) becomes <u>identically</u>

$$Z(n) = \sum_{k=0}^{N-1} \exp(-i2\pi kn/N)\, \tilde{z}(k) \quad \text{for all } n . \tag{12}$$

For N highly composite, FFT routines can be employed for efficient evaluation of (12) for $0 \leq n \leq N - 1$. The manipulation of (10) into (12) is called collapsing (or prealiasing), and the operation in (11) is modulo N addition. The nonzero values of $\{z_k\}$ in (10) can occur anywhere on the k-axis, and there can be an arbitrary number of them; nevertheless, (12) is an identity with (10). The value of $Z(n)$ for any n can be obtained immediately from the FFT output, by looking up the value in location n modulo N.

WAVEFORM CHARACTERIZATION AND RATE OF VARIATION

The continuous complex waveform of interest is s(t), with Fourier spectrum

$$S(f) = \int dt \exp(-i2\pi ft)\, s(t) \quad \text{for all } f . \tag{13}$$

We presume here that spectrum S(f) is bandlimited, with <u>total</u> extent F Hz; i.e.,

$$S(f) = 0 \quad \text{for } |f| > F/2 . \tag{14}$$

Notice in figure 1 that spectrum S(f) is centered at f = 0, without loss of generality, since waveform s(t) could be multiplied by $\exp(-i2\pi f_0 t)$ to downshift it by f_0 Hz, to any convenient center frequency, as desired.

If we were given a real waveform, we would replace it by its analytic waveform or complex envelope, thereby allowing the minimal possible time-sampling rate that can still exactly represent and recover the complex waveform. This sampling rate is half that required for sampling the corresponding real lowpass waveform, without loss of information. Nevertheless, the decreased sampling rate applied to the complex waveform is still sufficient to get an alias-free WDF and CAF. (Of course, the samples are now complex, whereas they were formerly real for the real waveform case.)
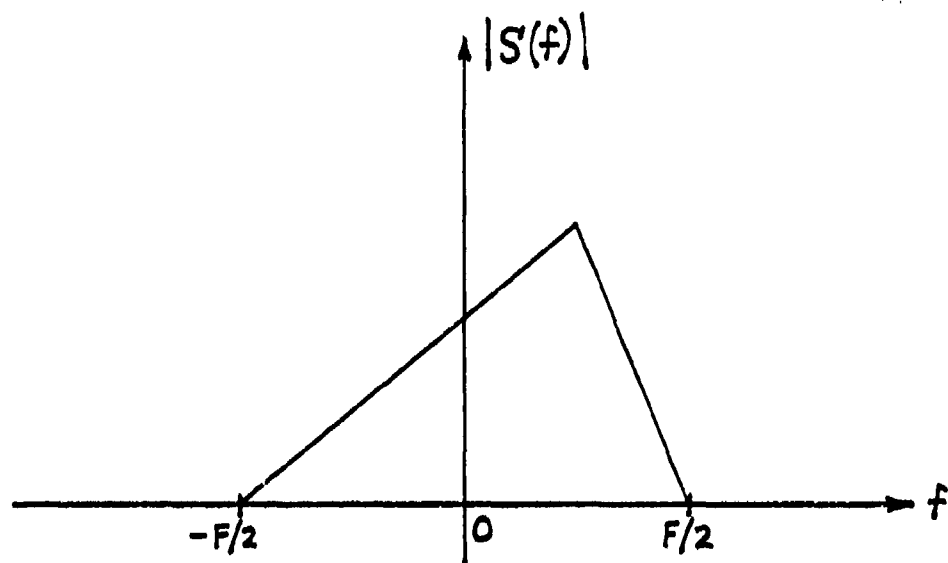
Figure 1.   Bandlimited Waveform Spectrum S(f)
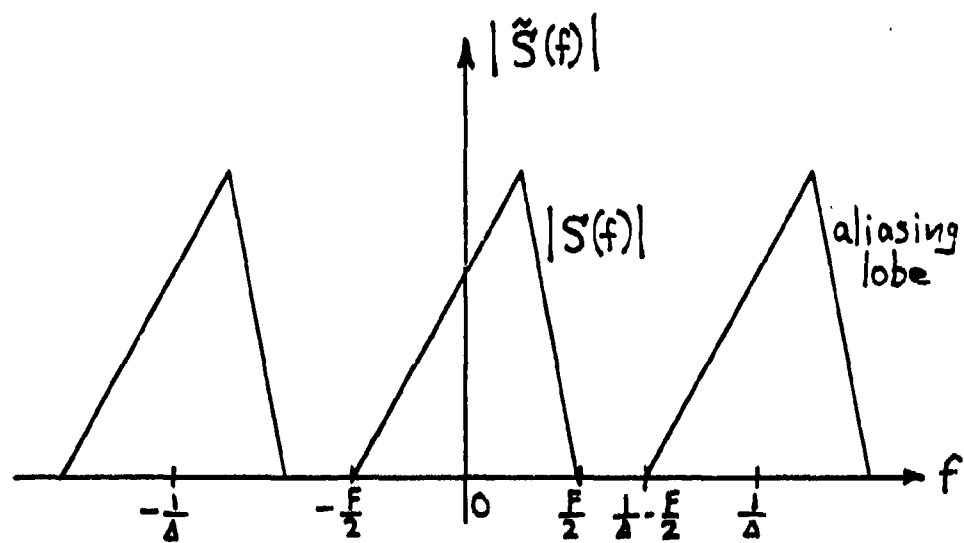
Figure 2.   Spectrum $\tilde{S}(f)$ of Sampled Waveform

SPECTRUM OF SAMPLED WAVEFORM

Waveform s(t) is sampled at time increment Δ seconds, yielding samples

$$\{s(k\Delta)\} \quad \text{for all integer } k \ . \tag{15}$$

The spectrum of this sampled waveform is defined by means of a Trapezoidal approximation to defining integral (13):

$$\tilde{S}(f) \equiv \Delta \sum_k \exp(-i2\pi f \Delta k) \ s(k\Delta) =$$

$$= \int dt \ \exp(-i2\pi ft) \ s(t) \ \Delta \sum_k \delta(t - k\Delta) =$$

$$= S(f) \circledast \sum_k \delta(f - \frac{k}{\Delta}) = \sum_k S(f - \frac{k}{\Delta}) \quad \text{for all } f \ , \tag{16}$$

where we used (6)-(9). The approximating spectrum $\tilde{S}(f)$ has period $1/\Delta$ in f and is depicted in figure 2. It will have nonoverlapping aliasing lobes if

$$\Delta < \frac{1}{F} \ . \tag{17}$$

This fundamental sampling rate condition will be presumed to be true, henceforth. In fact, in order to keep the number of samples $\{s(k\Delta)\}$ small, (17) will be presumed to be closely met. It is very important to minimize the number of samples that must be manipulated, so that the computational burden in evaluating the WDF is not overwhelming.

Another interpretation of approximation $\tilde{S}(f)$ is afforded by line 2 of (16): $\tilde{S}(f)$ is the spectrum (Fourier transform) of the signal s(t) sampled (multiplied) by the infinite impulse train at spacing Δ. This alternative interpretation will also arise later, when we investigate sampling relative to the WDF and its various two-dimensional transform domains.

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Since sampling rate condition (17) is presumed to be met, then spectrum $\widetilde{S}(f)$ in figure 2 can be gated with a rectangular function, and S(f) can be recovered; i.e.,

$$S(f) = \widetilde{S}(f) \; rect(f\Delta) \qquad for\ all\ f\ . \tag{18}$$

Therefore, the time waveform s(t) can also be recovered exactly, for all t, by means of inverse transform (5):

$$s(t) = \int df\ exp(i2\pi ft)\ \widetilde{S}(f)\ rect(f\Delta) =$$

$$= \widetilde{s}(t) \circledast \frac{1}{\Delta}\ sinc(\tfrac{t}{\Delta})\ . \tag{19}$$

However, since from line 2 of (16), product waveform

$$\widetilde{s}(t) = s(t)\ \Delta \sum_{k} \delta(t - k\Delta) = \Delta \sum_{k} s(k\Delta)\ \delta(t - k\Delta)\ , \tag{20}$$

then (19) becomes

$$s(t) = \sum_{k} s(k\Delta)\ sinc(\tfrac{t}{\Delta} - k) \qquad for\ all\ t\ , \tag{21}$$

which is the standard interpolation formula for a bandlimited waveform.

It should be pointed out here that (21) is not an attractive computational procedure, and that an excellent alternative is available. Namely, from (16), compute from the available samples,

$$\widetilde{S}(f) = \Delta \sum_{k} exp(-i2\pi f\Delta k)\ s(k\Delta) \qquad for\ |f| < \tfrac{1}{2\Delta}\ , \tag{22}$$

and then use the top line of (19) to recover waveform

$$s(t) = \int_{\frac{-1}{2\Delta}}^{\frac{1}{2\Delta}} df \, \exp(i2\pi ft) \, \tilde{S}(f) \qquad \text{for all } t .\qquad (23)$$

The reason this gating procedure is attractive is that (22) and (23) can both be done by FFT procedures.* Also, this seemingly trivial sidelight will reoccur in WDF and CAF reconstruction, where it will have a significant impact.

Notice that we have not defined a discrete spectrum, per se. Rather, we have concentrated on getting an approximation $\tilde{S}(f)$ to the original continuous spectrum $S(f)$, both defined for all f. If sampling condition (17) is met, $\Delta < 1/F$, the approximation affords the possibility of exact recovery of $S(f)$ at any f. This philosophy, namely avoiding arbitrary definitions of discrete functions, in favor of direct approximations to the desired continuous functions, is pursued throughout this report. It is believed that this clarifies the fundamental limitations and processing that must be performed in order to achieve the desired quantities. Finally, after demonstrating the viability of this approach, in order to reduce the mathematical equations to practical calculations, we discretize the time and/or frequency arguments of the approximations, as appropriate, and manipulate the equations into attractive FFT forms. We end up, of course, with discrete data processing forms that are suitable for efficient computer realization, but the

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*Actually, termination of the sum in (22) at finite k limits will yield an approximation to $\tilde{S}(f)$; the error can be controlled to any desired degree by taking enough terms. Also, the integral in (23) will have to be approximated, say, by the Trapezoidal rule; the attendant time-aliasing can be minimized by choosing the frequency increment small enough. These details will be investigated later.

discretization in time/frequency is deferred to the latest possible location, since it is not fundamental to the ideas of controlling or eliminating aliasing.

The sampling increment Δ will not be set equal to 1 in this report, for several reasons. It is easier to keep track of dimensions, and dimensional checks on the equations are accomplished more readily. It is also easier to obtain physical interpretation of time instants and increments, as well as frequency limits and bandwidths. Finally, it will be seen to eliminate confusion and ambiguity as to precisely where time and frequency samples of the temporal correlation function, WDF, and CAF are being taken; the importance of this last point can not be overemphasized.

EXAMPLES

It is very informative at this point to consider a couple of continuous waveforms and their corresponding WDFs, in terms of their rates of variation. Consider first, spectrum

$$S(f) = \frac{1}{F} \operatorname{rect}(\frac{f}{F}) = \begin{cases} 1/F & \text{for } |f| < F/2 \\ 0 & \text{otherwise} \end{cases}, \tag{24}$$

for which the waveform is

$$s(t) = \operatorname{sinc}(Ft) = \frac{\sin(\pi Ft)}{\pi Ft} \quad \text{for all } t. \tag{25}$$

The corresponding WDF, at time t and frequency f, is [9, (10)]

$$W(t,f) = \int d\tau \, \exp(-i2\pi f\tau) \, s(t + \tfrac{\tau}{2}) \, s^*(t - \tfrac{\tau}{2}) =$$

$$= \int d\upsilon \, \exp(i2\pi\upsilon t) \, S(f + \tfrac{\upsilon}{2}) \, S^*(f - \tfrac{\upsilon}{2}) =$$

$$= \left\{ \begin{array}{ll} \dfrac{\sin[2\pi Ft(1 - 2|f|/F)]}{\pi F^2 t} & \text{for } |f| < \dfrac{F}{2} \\[3ex] 0 & \text{for } |f| > \dfrac{F}{2} \end{array} \right\} \text{ for all } t$$

$$= \frac{2}{F}\left(1 - 2\,\frac{|f|}{F}\right)\text{sinc}\,[2Ft(1 - 2\cdot\tfrac{|f|}{F})] \quad \text{rect}(\tfrac{f}{F}) \ . \tag{26}$$

Then, for instance, the slice of the WDF at zero frequency,

$$W(t,0) = \frac{2}{F} \text{ sinc}(2Ft) \ , \tag{27}$$

varies _twice_ as fast as waveform $s(t)$ in (25). Therefore, although sampling $s(t)$ in (25) with time increment $\Delta < 1/F$ is sufficient to reconstruct $s(t)$, we need a time increment half as large in order to adequately sample slice (27) of the WDF at $f = 0$. In fact, $W(t,f)$ in (26) varies faster with t then $s(t)$ does, whenever $|f| < F/4$.

This example points out that the WDF must be computed _twice_ as finely as the waveform samples, lest important information about the energy distribution of $s(t)$ in t,f space be lost. In fact, if (27) were computed at time points

$$t_n = (n + \tfrac{1}{2})\,\frac{1}{F} \quad \text{for all } n \ , \tag{28}$$

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which have time increment $\Delta_t = 1/F$, then the WDF values obtained would be

$$W\left(\left(n + \tfrac{1}{2}\right) \tfrac{1}{F}, 0\right) = \tfrac{2}{F} \text{sinc}(2n + 1) = 0 \qquad \text{for all } n. \tag{29}$$

We would be led to believe from samples (29) that there is no energy along the $f = 0$ line in t,f space, whereas continuous version (27) indicates a considerable contribution.

A second example is

$$s(t) = \exp\left(-\frac{t^2}{2\sigma^2}\right),$$

$$S(f) = \sqrt{2\pi}\, \sigma\, \exp(-2\pi^2\sigma^2 f^2), \tag{30}$$

for which the WDF is

$$W(t,f) = 2\sqrt{\pi}\,\sigma\, \exp\left(-\frac{t^2}{\sigma^2} - 4\pi^2\sigma^2 f^2\right). \tag{31}$$

This WDF varies faster with t than $s(t)$ does, and faster with f than $S(f)$ does. In fact, the rates of variation of $W(t,f)$ and $|s(t)|^2$ are the same with t, while those of $W(t,f)$ and $|S(f)|^2$ are the same in f.

Both of the examples above illustrate the need to compute the WDF at finer increments than are adequate for the time waveform or spectrum. However, this does not mean that the time waveform need be sampled more frequently than requirement (17). Rate (17) is fine for sampling waveform $s(t)$, but the corresponding WDF can and must then be computed at finer increments.

TWO-DIMENSIONAL CONTINUOUS FUNCTIONS

For a continuous waveform s(t) with spectrum S(f), there are four useful two-dimensional characterizations. The first is the continuous temporal correlation function (TCF)

$$R(t,\tau) = s(t + \tfrac{\tau}{2})\, s^{*}(t - \tfrac{\tau}{2}) \quad \text{for all } t,\tau . \qquad (32)$$

Variable t is absolute time in seconds, while τ is relative time or time separation. The corresponding Wigner distribution function (WDF) is a Fourier transform on τ:

$$W(t,f) = \int d\tau \, \exp(-i2\pi f\tau)\, R(t,\tau) \quad \text{for all } t,f . \qquad (33)$$

The alternative Fourier transform on t yields the complex ambiguity function (CAF):

$$\chi(\upsilon,\tau) = \int dt \, \exp(-i2\pi\upsilon t)\, R(t,\tau) \quad \text{for all } \upsilon,\tau . \qquad (34)$$

Functions W and χ are two-dimensional Fourier transforms of each other.

Finally, completing both routes (by t or by τ), we have the spectral correlation function (SCF) as another Fourier transform, according to several equivalent forms

$$A(\upsilon,f) = \int dt \ \exp(-i2\pi\upsilon t) \ W(t,f) =$$

$$= \int d\tau \ \exp(-i2\pi f\tau) \ \chi(\upsilon,\tau) =$$

$$= \iint dt \ d\tau \ \exp(-i2\pi\upsilon t - i2\pi f\tau) \ R(t,\tau) =$$

$$= S(f + \tfrac{\upsilon}{2}) \ S^*(f - \tfrac{\upsilon}{2}) \qquad \text{for all } \upsilon,f \ . \tag{35}$$

This last relation in terms of the spectrum S(f) of waveform s(t) will turn out to be extremely important and useful. It also enables interpretation of f as absolute frequency in Hz, while υ is relative frequency or frequency separation.

The names for the TCF and SCF have been drawn from the similarity of their forms in (32) and (35), respectively, to correlation operations. The latter name is also used in [10, (5)-(7)] for a similar quantity.

Recalling the bandlimited character of S(f) in (14) and figure 1, we see that SCF A(υ,f) in (35) can be nonzero only when

$$\left| f \pm \tfrac{\upsilon}{2} \right| < \tfrac{F}{2} \ . \tag{36}$$

This region in the two-frequency domain (υ,f plane) is depicted in figure 3. It is a diamond-shaped region centered at the origin of the υ,f plane. Outside this diamond, SCF A(υ,f) is identically zero. Thus, a bandlimited spectrum S(f) is reflected in the υ,f domain as a diamond-limited SCF A(υ,f).

Since (35) can be inverted to give

$$W(t,f) = \int d\upsilon \, \exp(i2\pi\upsilon t) \, A(\upsilon,f) \, ,$$

$$x(\upsilon,\tau) = \int df \, \exp(i2\pi f\tau) \, A(\upsilon,f) \, , \tag{37}$$

it follows from figure 3 that

$$W(t,f) = 0 \quad \text{for } |f| > F/2 \, ,$$

$$x(\upsilon,\tau) = 0 \quad \text{for } |\upsilon| > F \, . \tag{38}$$

Thus the WDF and CAF are bandlimited in their respective frequency variables. These properties will be useful later when we study the effects of aliasing in the various domains. More generally, the extents and rates of variation of the TCF, WDF, CAF, and SCF are summarized in appendix A.

The following symmetry properties on the TCF and SCF reduce computational effort by a factor of two:

$$R(t,-\tau) = R^{*}(t,\tau) \, ,$$

$$A(-\upsilon,f) = A^{*}(\upsilon,f) \, .$$

These follow immediately from (32) and (35), respectively.
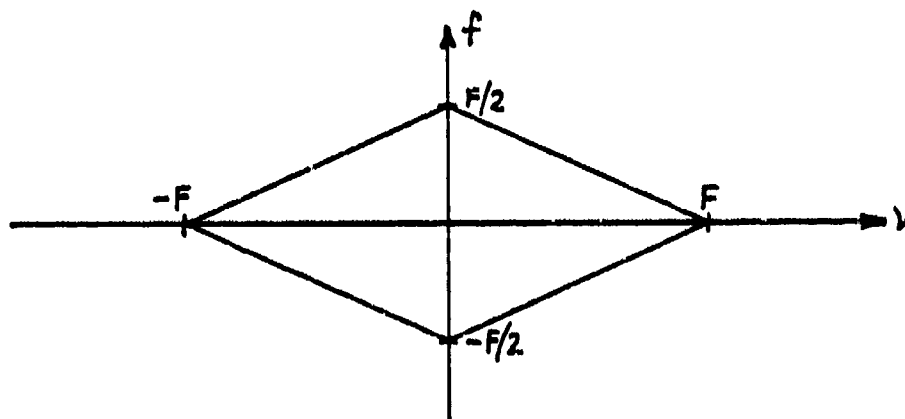

Figure 3.   Extent of Spectral Correlation Function $A(\upsilon,f)$

TWO-DIMENSIONAL FUNCTIONS FOR DISCRETE-TIME SAMPLES

The available data samples of waveform $s(t)$ are, as given in (15),

$$\left\{s(k\Delta)\right\} \quad \text{for integer } k \; .$$

SAMPLED TEMPORAL CORRELATION FUNCTION

From these values, the totality of information, that can be computed regarding the continuous TCF $R(t,\tau)$ in (32), are the two sets of discrete values

$$R(m\Delta, 2q\Delta) = s(m\Delta + q\Delta) \; s^*(m\Delta - q\Delta) \qquad (39a)$$

and

$$R\left(\left(m + \frac{1}{2}\right)\Delta, (2q + 1)\Delta\right) = s(m\Delta + q\Delta + \Delta) \; s^*(m\Delta - q\Delta) \qquad (39b)$$

for integers m and q.

Thus both the t and τ variables in $R(t,\tau)$ are discretized, as indicated in figure 4. However, observe that the available information is interspersed in the t,τ plane. Thus, for fixed t, the separation in available τ values is 2Δ, not Δ, whether t/Δ is integer or half-integer. Similarly, for fixed τ, the separation in available t values is Δ, not $\Delta/2$, whether τ/Δ is an even or odd integer. This lack of intermediate values in both slices is what has led (in the past) to incomplete processing of the available information. What is needed to solve the aliasing problem is a combination of all the interspersed information in figure 4 into a single unified two-dimensional description. That solution will be found to reside in the SCF domain, ν, f.
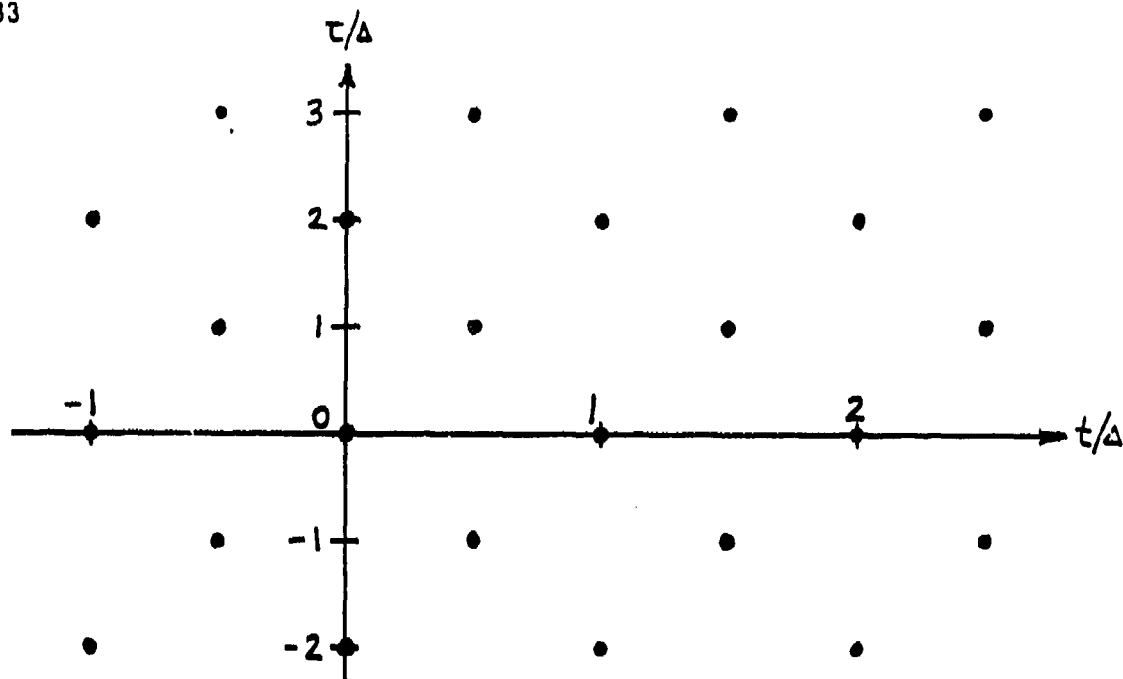
Figure 4. Available Values of TCF $R(t,\tau)$

APPROXIMATE WIGNER DISTRIBUTION FUNCTION

Guided by definition (33), we adopt the following Trapezoidal approximation to the continuous WDF $W(t,f)$ at time $t = m\Delta$, m integer:

$$W_a(m\Delta,f) \equiv 2\Delta \sum_q \exp(-i2\pi f2q\Delta) \, R(m\Delta,2q\Delta) \qquad \text{for all } f \,. \qquad (40)$$

Subscript a on W_a denotes that it is only an approximation to the true continuous W. Notice that the τ increment in (40) is 2Δ, as it must be, according to (39a) and figure 4; we are taking a vertical slice at $t = m\Delta$ in figure 4. The approximation in (40) is always real. It utilizes only the upper line of information available in (39). Notice also that this function is defined for all f.

However, there is an **additional** approximation to $W(t,f)$ available at time $t = (m + \frac{1}{2})\Delta$, by use of the bottom line of (39); namely, guided again

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by definition (33), we have Trapezoidal approximation

$$W_a\left((m + \tfrac{1}{2})\Delta, f\right) \cong 2\Delta \sum_q \exp[-i2\pi f(2q + 1)\Delta]\, R\left((m + \tfrac{1}{2})\Delta,\, (2q + 1)\Delta\right) \text{ for all } f \,. \tag{41}$$

m is still an integer. Again, the τ increment is 2Δ, but it is shifted by Δ, as (39b) and figure 4 dictate. We are now taking a vertical slice at $t = (m + \tfrac{1}{2})\Delta$ in figure 4; this is in keeping with the philosophy developed earlier in (24)-(31). Approximation (41) is also real.

Equations (40) and (41) can be developed into some informative forms; from (40) and (9), approximation

$$W_a(m\Delta, f) = \int d\tau \, \exp(-i2\pi f\tau)\, R(m\Delta, \tau)\, 2\Delta \sum_q \delta(\tau - 2q\Delta) =$$

$$= W(m\Delta, f) \overset{f}{\circledast} \sum_q \delta(f - \tfrac{q}{2\Delta}) =$$

$$= \sum_q W(m\Delta, f - \tfrac{q}{2\Delta}) \quad \text{for all } f \,. \tag{42}$$

Similarly, (41) yields

$$W_a\left((m + \tfrac{1}{2})\Delta, f\right) = \int d\tau \, \exp(-i2\pi f\tau)\, R\left((m + \tfrac{1}{2})\Delta, \tau\right)\, 2\Delta \sum_q \delta(\tau - 2q\Delta - \Delta) =$$

$$= W\left((m + \tfrac{1}{2})\Delta, f\right) \overset{f}{\circledast} \sum_q (-1)^q \delta(f - \tfrac{q}{2\Delta}) =$$

$$= \sum_q (-1)^q W\left((m + \tfrac{1}{2})\Delta,\, f - \tfrac{q}{2\Delta}\right) \quad \text{for all } f \,. \tag{43}$$

The $(-1)^q$ factor is due to the time delay of Δ seconds in the impulse train; more generally, for delay τ_0,

$$\int d\tau \, \exp(-i2\pi f\tau) \, 2\Delta \sum_q \delta(\tau - 2q\Delta - \tau_0) =$$

$$= \exp(-i2\pi f\tau_0) \sum_q \delta(f - \frac{q}{2\Delta}) =$$

$$= \sum_q \exp(-i\pi q \, \tau_0/\Delta) \, \delta(f - \frac{q}{2\Delta}) . \tag{44}$$

The two relations, (42) and (43), are equivalent to those given in [4, (9)] and [6, (14)]. Observe that the aliasing lobes are separated by only $(2\Delta)^{-1}$ in the frequency domain, not $1/\Delta$ as was the case for the signal aliasing lobes in figure 2. These approximations, W_a, are illustrated in figure 5, for two adjacent time instants at $m\Delta$ and $(m + \frac{1}{2})\Delta$. We have used property (38) in drawing figure 5.

In order for either approximation, by itself, to be free of aliasing, we would need

$$\frac{F}{2} < \frac{1}{2\Delta} - \frac{F}{2} , \quad i.e., \quad \Delta < \frac{1}{2F} . \tag{45}$$

This relation, obtained directly from both plots in figure 5, is the usual one quoted[*] regarding an alias-free WDF. It is seen to require a sampling rate twice as fine as (17). If we satisfy (17), but not (45), then the approximations in figure 5 are significantly aliased.

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[*] The case where s(t) is a real waveform is treated in appendix A.

$W_a(m\Delta, f)$

$W(m\Delta, f)$

aliasing lobe

$-\frac{1}{2\Delta}$ $-\frac{F}{2}$ 0 $\frac{1}{2\Delta}-\frac{F}{2}$ $\frac{F}{2}$ $\frac{1}{2\Delta}$ f

$W_a((m+\frac{1}{2})\Delta, f)$

$W((m+\frac{1}{2})\Delta, f)$

$\frac{-1}{2\Delta}$ $-\frac{F}{2}$ 0 $\frac{1}{2\Delta}-\frac{F}{2}$ $\frac{F}{2}$ $\frac{1}{2\Delta}$ f

aliasing lobe
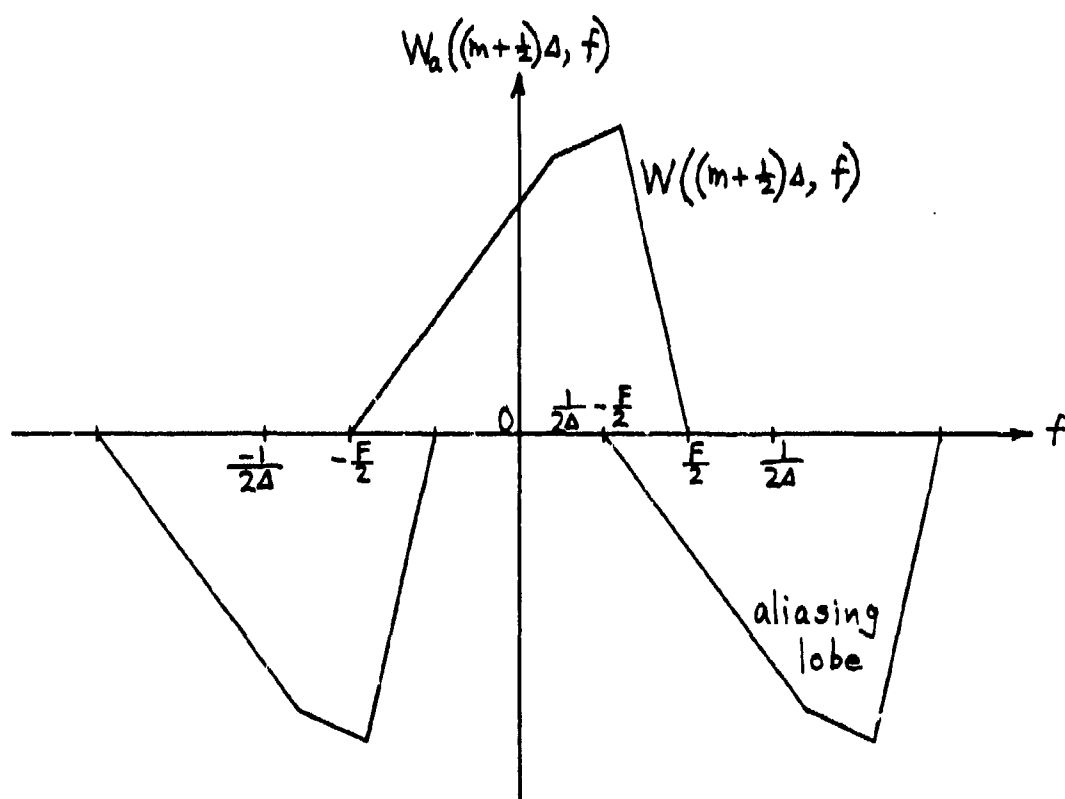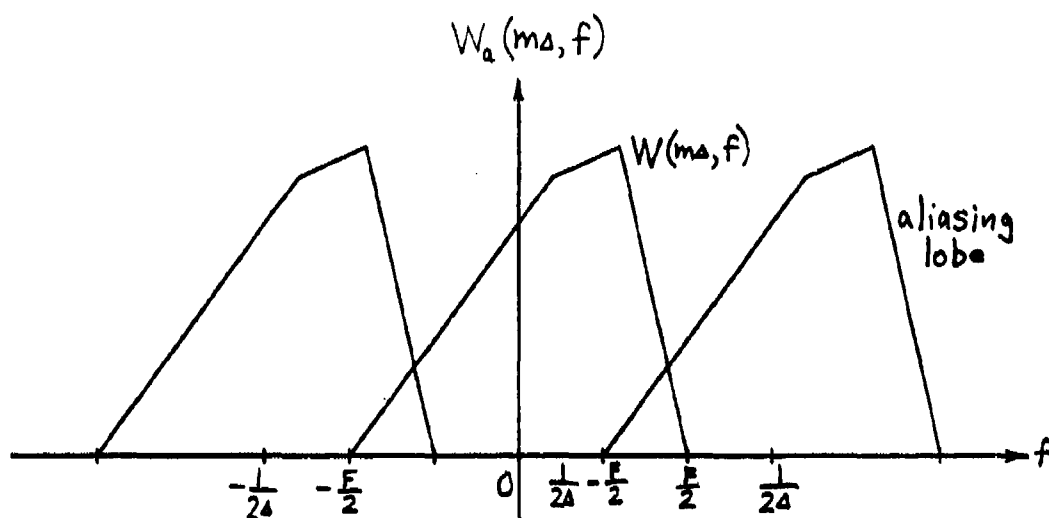
Figure 5. WDF Approximations

The two approximations, (40) and (41), use all the available information (39) about the TCF $R(t,\tau)$ in the t,τ plane. However, we cannot average these two WDF approximations, in hopes of cancelling out the close-in lobes centered at $f = \pm(2\Delta)^{-1}$, because the two times, $m\Delta$ and $(m + \frac{1}{2})\Delta$, are not identical. (This timing observation is one reason for keeping Δ itself in all the equations, rather than setting Δ equal to 1 and losing track of the meaning of m vs. $m + \frac{1}{2}$). Nor can we discard either one of approximations (42) and (43), especially if criterion $\Delta < 1/F$ is closely met; the examples in (24)-(31) amply demonstrate the rapid variation of the WDF with time t.

We see from (40) and (41) that approximations to the continuous WDF are available at discrete time values with separation $\Delta_t = \Delta/2$ and at a continuum of f values. Thus we have succeeded in eliminating the discrete nature of one of the two initial time variables in the TCF, namely τ.

APPROXIMATE SPECTRAL CORRELATION FUNCTION VIA WDF

Guided by line 1 of definition (35), we adopt the following Trapezoidal approximation to the original SCF $A(\nu, f)$:

$$A_a^{(w)}(\nu, f) \equiv \frac{\Delta}{2} \sum_n \exp\left(-i2\pi\nu \frac{n\Delta}{2}\right) W_a\left(\frac{n\Delta}{2}, f\right) =$$

$$= \frac{\Delta}{2}\left\{\sum_{n \text{ even}} + \sum_{n \text{ odd}}\right\} \exp(-i2\pi\nu \frac{n\Delta}{2}) W_a(\frac{n\Delta}{2}, f) \qquad \text{for all } \nu, f . \qquad (46)$$

Notice that this function is defined on a continuum in ν, f space. The superscript w on approximation A_a denotes the fact that we have used the WDF route to get into the ν, f plane. The increment on t in (46) is $\Delta_t = \Delta/2$, in keeping with the available WDF values in (40) and (41) together. This approximation satisfies the symmetry rule, $A_a^{(w)}(-\nu, f) = A_a^{(w)}(\nu, f)^*$, just as for the original SCF $A(\nu, f)$.

We let $n = 2m$ in the first sum in (46), and let $n = 2m + 1$ in the second sum. There follows, upon use of (42) and (43),

$$A_a^{(w)}(\nu, f) = \frac{\Delta}{2} \sum_m \exp(-i2\pi\nu\Delta m) W_a(m\Delta, f) +$$

$$+ \frac{\Delta}{2} \sum_m \exp\left(-i2\pi\nu\Delta(m + \tfrac{1}{2})\right) W_a\left((m + \tfrac{1}{2})\Delta, f\right) =$$

$$= \frac{1}{2} \sum_q \Delta \sum_m \exp(-i2\pi\nu\Delta m) W(m\Delta, f - \tfrac{q}{2\Delta}) +$$

$$+ \frac{1}{2} \sum_q (-1)^q \Delta \sum_m \exp[-i2\pi\nu\Delta(m + \tfrac{1}{2})] W\left((m + \tfrac{1}{2})\Delta, f - \tfrac{q}{2\Delta}\right) . \qquad (47)$$

But

$$\Delta \sum_m \exp(-i2\pi\upsilon\Delta m) \ W(m\Delta,f') \ =$$

$$= \int dt \ \exp(-i2\pi\upsilon t) \ W(t,f') \ \Delta \sum_m \delta(t - m\Delta) \ =$$

$$= A(\upsilon,f') \ \overset{\upsilon}{\circledast} \ \sum_m \delta(\upsilon - \frac{m}{\Delta}) \ =$$

$$= \sum_m A(\upsilon - \frac{m}{\Delta},f') \ , \qquad\qquad (48)$$

while

$$\Delta \sum_m \exp[-i2\pi\upsilon\Delta(m + \tfrac{1}{2})] \ W\left((m + \tfrac{1}{2})\Delta,f'\right) \ =$$

$$= \int dt \ \exp(-i2\pi\upsilon t) \ W(t,f') \ \Delta \sum_m \delta\left(t - (m + \tfrac{1}{2})\Delta\right) \ =$$

$$= A(\upsilon,f') \ \overset{\upsilon}{\circledast} \ \sum_m (-1)^m \delta(\upsilon - \frac{m}{\Delta}) \ =$$

$$= \sum_m (-1)^m \ A(\upsilon - \frac{m}{\Delta},f') \ . \qquad\qquad (49)$$

The use of these two relations in (47) yields the approximate SCF

$$A_a^{(w)}(v,f) = \frac{1}{2} \sum_q \sum_m A(v - \frac{m}{\Delta}, f - \frac{q}{2\Delta}) + \qquad (50a)$$

$$+ \frac{1}{2} \sum_q \sum_m (-1)^{q+m} A(v - \frac{m}{\Delta}, f - \frac{q}{2\Delta}) = \qquad (50b)$$

$$= \sum_{\substack{q \\ q+m \text{ even}}} \sum_m A(v - \frac{m}{\Delta}, f - \frac{q}{2\Delta}) \quad \text{for all } v,f . \qquad (51)$$

We now have a function defined on a two-dimensional continuum in the two-frequency domain, v,f.

At this point, the reason for pursuing the use of <u>all</u> the available information becomes obvious. All the close-in lobes that caused problems have precisely cancelled in the SCF domain! Figure 6 depicts the regions in the v,f plane where approximation $A_a^{(w)}(v,f)$ in (51) can be nonzero; see also figure 3.

The SCF term in (50a) corresponds to use of only the information about the TCF given in (39a), within a factor of 2, whereas (50b) arises from (39b). Term (50a) by itself contains all the aliasing lobes centered at $v = \frac{m}{\Delta}, f = \frac{q}{2\Delta}$, with separations $\Delta_v = 1/\Delta$, $\Delta_f = (2\Delta)^{-1}$; condition (17) is then insufficient to prevent overlap, and (50a) is seriously aliased. A similar situation exists for (50b) by itself. It is only the average of these two pieces of information that succeeds in elimination of the troublesome close-in aliasing lobes in v,f space. And it is only in this last domain, where the functions are continuous in both variables, that this average can be conducted.

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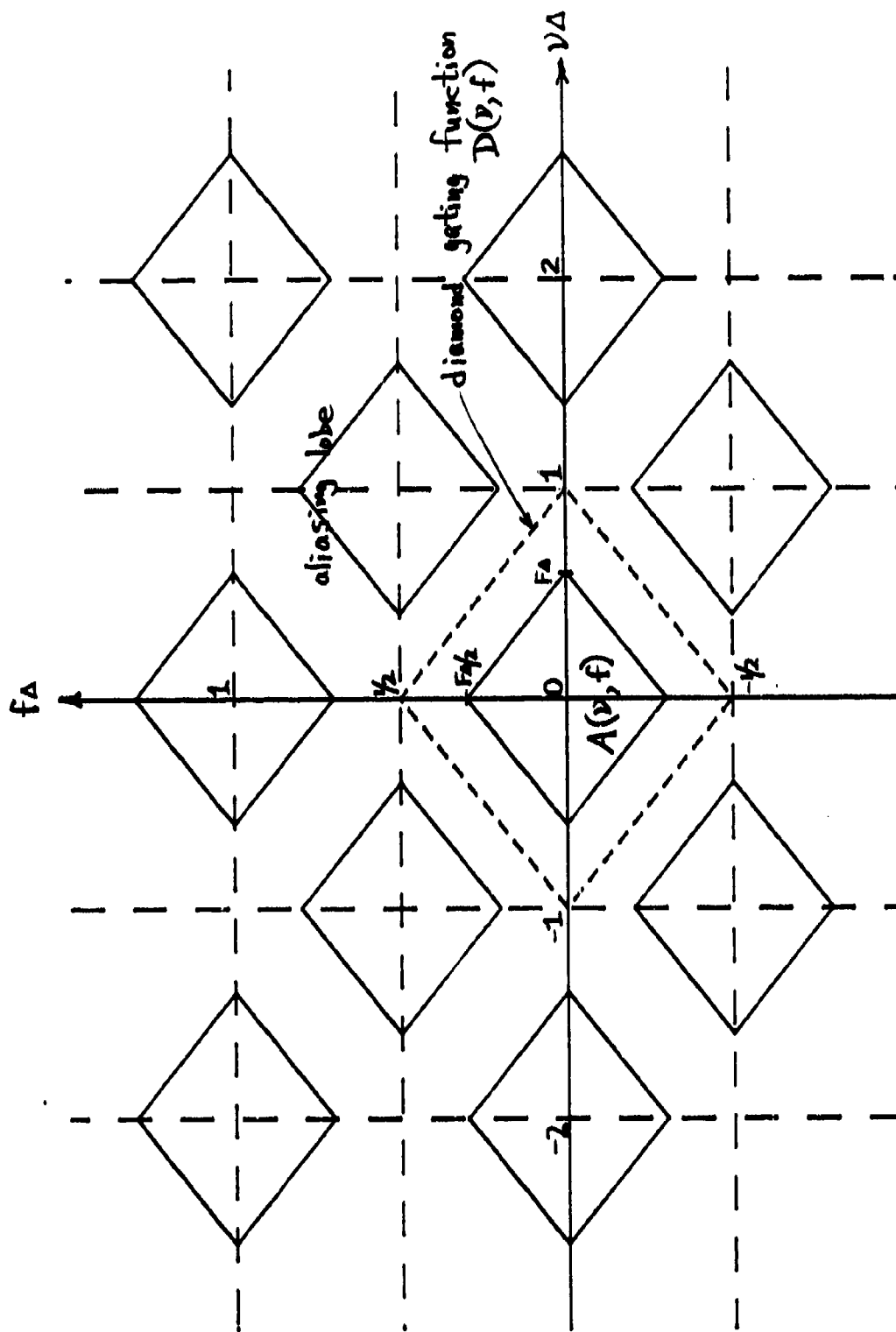
Figure 6. Approximate SCF $A_a(\nu, f)$

There will be no overlap of any of the remaining aliasing lobes in figure 6 if we choose, as in (17),

$$\Delta < \frac{1}{F} \ . \tag{52}$$

Notice that we do <u>not</u> have to require $\Delta < (2F)^{-1}$, in order to avoid the overlap. Furthermore, if we define the diamond gating function (see figure 6)

$$D(\upsilon,f) = \begin{cases} 1 & \text{for } \left|f \pm \frac{\upsilon}{2}\right| < \frac{1}{2\Delta} \\ 0 & \text{otherwise} \end{cases} , \tag{53}$$

then we can recover exactly the original SCF from approximation (51) according to

$$A_a^{(w)}(\upsilon,f) \ D(\upsilon,f) = A(\upsilon,f) \qquad \text{for all } \upsilon,f \ . \tag{54}$$

But recovery of $A(\upsilon,f)$ is tantamount to recovering the exact continuous WDF, since

$$W(t,f) = \int d\upsilon \ \exp(i2\pi\upsilon t) \ A(\upsilon,f) \qquad \text{for all } t,f \ . \tag{55}$$

Thus, criterion (52) is sufficient to guarantee the possibility of getting an alias-free WDF from discrete-time data.

Additional interpretations of (54) and a simple method of computing approximation $A_a^{(w)}(\upsilon,f)$ are addressed in the next section, after we have also looked at the route to the υ,f plane by way of the CAF.

APPROXIMATE COMPLEX AMBIGUITY FUNCTION

Based on definition (34), we utilize the Trapezoidal approximation to the continuous CAF $x(v,\tau)$ at delay $\tau = 2q\Delta$, q integer:

$$x_a(v,2q\Delta) \equiv \Delta \sum_m \exp(-i2\pi v \Delta m) \, R(m\Delta, 2q\Delta) =$$

$$= \int dt \, \exp(-i2\pi vt) \, R(t, 2q\Delta) \, \Delta \sum_m \delta(t - m\Delta) =$$

$$= x(v, 2q\Delta) \overset{v}{\circledast} \sum_m \delta\left(v - \frac{m}{\Delta}\right) =$$

$$= \sum_m x\left(v - \frac{m}{\Delta}, \, 2q\Delta\right) \qquad \text{for all } v \,. \tag{56}$$

Notice that the t increment is Δ, as it must be, according to (39a) and figure 4; we are taking a horizontal slice at $\tau = 2q\Delta$ in figure 4.

However, there is an _additional_ approximation available to the CAF, at delay $\tau = (2q + 1)\Delta$; again, referring to (34),

$$x_a\left(v,(2q + 1)\Delta\right) \equiv \Delta \sum_m \exp\left[-i2\pi v \Delta\left(m + \tfrac{1}{2}\right)\right] R_a\left(\left(m + \tfrac{1}{2}\right)\Delta, (2q + 1)\Delta\right) =$$

$$= \int dt \, \exp(-i2\pi vt) \, R\left(t, (2q + 1)\Delta\right) \Delta \sum_m \delta\left(t - (m + \tfrac{1}{2})\Delta\right) =$$

$$= x\left(v, (2q + 1)\Delta\right) \overset{v}{\circledast} \sum_m (-1)^m \, \delta\left(v - \frac{m}{\Delta}\right) =$$

$$= \sum_m (-1)^m \, x\left(v - \frac{m}{\Delta}, (2q + 1)\Delta\right) \qquad \text{for all } v \,. \tag{57}$$

The t increment is again Δ, but it is shifted by $\Delta/2$, in keeping with (39b) and figure 4. We are now taking a horizontal slice at $\tau = (2q + 1)\Delta$ in figure 4. The $(-1)^m$ factor is explained by (44).

The aliasing lobes in (56) and (57) are separated only by $\Delta_\nu = 1/\Delta$ and will overlap on the ν axis unless $\Delta < (2F)^{-1}$; see (38) and figure 7. Thus, the approximate CAF, $\chi_a(\nu, n\Delta)$ for n integer, suffers overlap due to aliasing, just as the WDF, $W_a(\frac{n}{2}\Delta, f)$ for n integer, does; elimination of overlap is achieved only if the stringent requirement $\Delta < (2F)^{-1}$ is met. Furthermore, again, we cannot directly average the two results in figure 7, in hopes of canceling the close-in lobes centered at $\nu = \pm 1/\Delta$, because the two delays, $2q\Delta$ and $(2q + 1)\Delta$, are not identical.

Equations (56) and (57), together, illustrate that approximations to the continuous CAF are available at discrete delay values with separation $\Delta_\tau = \Delta$ and at a continuum of ν values. Now we have succeeded in eliminating the discrete nature of the other of the two initial time variables in the TCF, namely t. The remaining Fourier transform into the SCF domain will eliminate the other discrete variable.
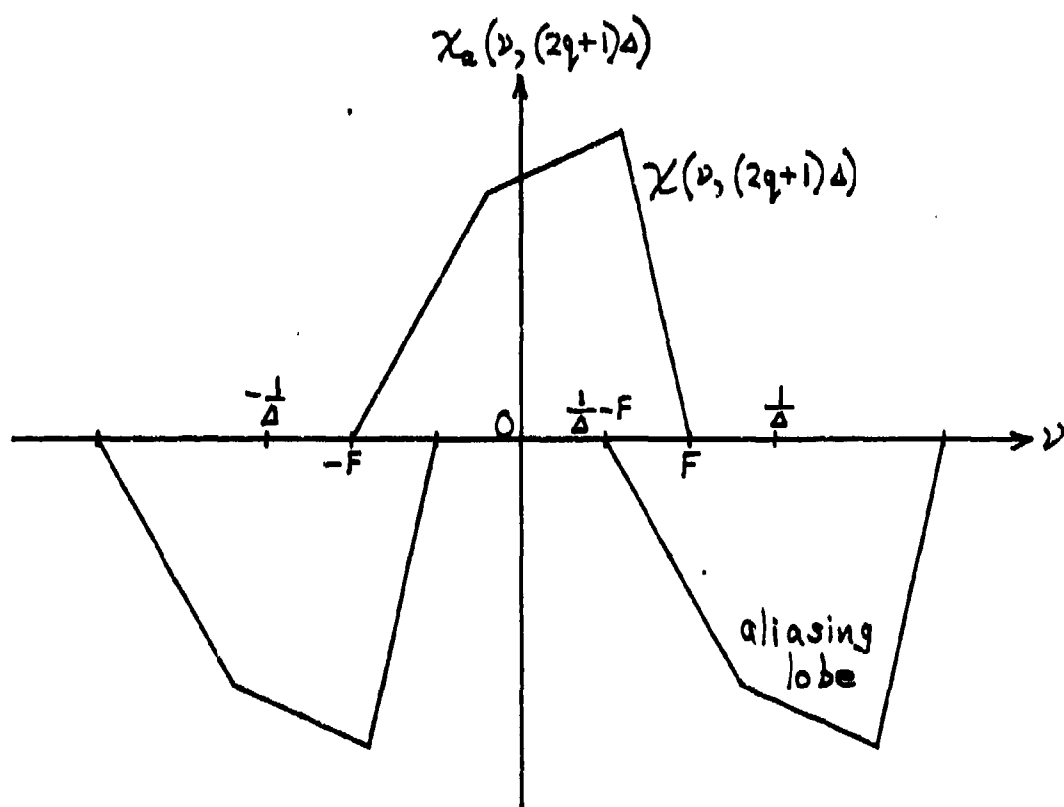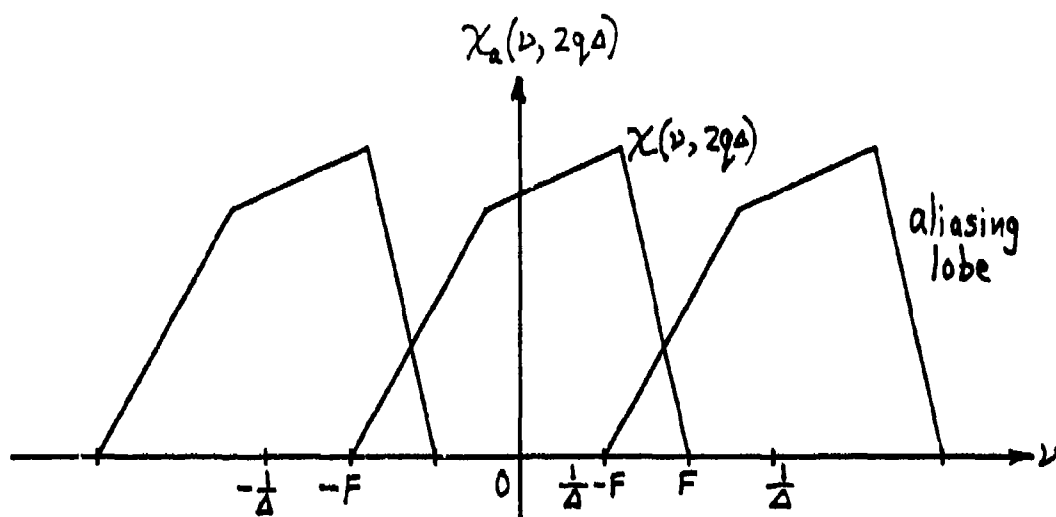
Figure 7. CAF Approximations

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APPROXIMATE SPECTRAL CORRELATION FUNCTION VIA CAF .

Guided by line 2 of definition (35), we obtain the following approximation to the original SCF $A(v,f)$:

$$A_a^{(c)}(v,f) \equiv \Delta \sum_n \exp(-i2\pi fn\Delta) \; \chi_a(v,n\Delta) =$$

$$= \Delta \left\{ \sum_{n \text{ even}} + \sum_{n \text{ odd}} \right\} \exp(-i2\pi fn\Delta) \; \chi_a(v,n\Delta) \quad \text{for all } v,f . \quad (58)$$

The superscript c on approximation A_a denotes that we are obtaining this result by way of the CAF. The increment on τ in (58) is $\Delta_\tau = \Delta$, in keeping with the available CAF values in (56) and (57) together.

Let $n = 2q$ in the first sum in (58), and let $n = 2q + 1$ in the second sum. There follows, upon use of (56) and (57),

$$A_a^{(c)}(v,f) = \Delta \sum_q \exp(-i2\pi f\Delta 2q) \; \chi_a(v,2q\Delta) +$$

$$+ \Delta \sum_q \exp[-i2\pi f\Delta(2q + 1)] \; \chi_a\left(v,(2q + 1)\Delta\right) =$$

$$= \sum_m \Delta \sum_q \exp(-i2\pi f\Delta 2q) \; \chi\left(v - \frac{m}{\Delta}, \; 2q\Delta\right) +$$

$$+ \sum_m (-1)^m \Delta \sum_q \exp[-i2\pi f\Delta(2q + 1)] \; \chi\left(v - \frac{m}{\Delta}, \; (2q + 1)\Delta\right) . \quad (59)$$

But

$$\Delta \sum_{q} \exp(-i2\pi f\Delta 2q) \; x(\upsilon',2q\Delta) \; =$$

$$= \int d\tau \; \exp(-i2\pi f\tau) \; x(\upsilon',\tau) \; \Delta \sum_{q} \delta(\tau - 2q\Delta) \; =$$

$$= A(\upsilon',f) \overset{f}{\circledast} \frac{1}{2} \sum_{q} \delta\!\left(f - \frac{q}{2\Delta}\right) \; =$$

$$= \frac{1}{2} \sum_{q} A\!\left(\upsilon',f - \frac{q}{2\Delta}\right) \; , \tag{60}$$

while

$$\Delta \sum_{q} \exp[-i2\pi f\Delta(2q + 1)] \; x\!\left(\upsilon',(2q + 1)\Delta\right) \; =$$

$$= \int d\tau \; \exp(-i2\pi f\tau) \; x(\upsilon',\tau) \; \Delta \sum_{q} \delta\!\left(\tau - (2q + 1)\Delta\right) \; =$$

$$= A(\upsilon',f) \overset{f}{\circledast} \frac{1}{2} \sum_{q} (-1)^{q} \; \delta\!\left(f - \frac{q}{2\Delta}\right) \; =$$

$$= \frac{1}{2} \sum_{q} (-1)^{q} \; A\!\left(\upsilon',f - \frac{q}{2\Delta}\right) \; . \tag{61}$$

The use of these two relations in (59) yields the approximate SCF

$$A_a^{(c)}(\upsilon,f) = \frac{1}{2} \sum_m \sum_q A(\upsilon - \frac{m}{\Delta}, f - \frac{q}{2\Delta}) +$$

$$+ \frac{1}{2} \sum_m \sum_q (-1)^{m+q} A(\upsilon - \frac{m}{\Delta}, f - \frac{q}{2\Delta}) =$$

$$= \sum_{\substack{m \\ m+q \text{ even}}} \sum_q A(\upsilon - \frac{m}{\Delta}, f - \frac{q}{2\Delta}) \quad \text{for all } \upsilon, f \ . \qquad (62)$$

But this result is identical to the approximate SCF $A_a^{(w)}(\upsilon,f)$ given in (51) and figure 6. That is, we obtain the same continuous approximation in the υ,f domain, whether we approach it via the WDF or the CAF. This apparently fortuitous result is due to the fact that we used all the available information about the TCF when we started with (39), and kept all of it in passage through the WDF or CAF domains.

Figure 6 is again applicable, and we now see that we can drop superscripts w and c from (51) and (62), respectively, since there is only one approximation in the SCF domain. (The comments following (51) are also directly applicable here.)

A rigorous proof of the equality of the two approximations available for the SCF is given in appendix B. It utilizes an impulsive sampling approach, similar to (9) but in two dimensions, and can be considered as an alternative to the approximation approach developed in this section. Of course, the end result for the SCF in the υ,f domain is again (51) or (62) or figure 6.

SUMMARY STATUS IN ALL FOUR DOMAINS

The results for the approximations to the TCF, WDF, CAF, and SCF are sketched in figure 8. These plots are a condensation of the exact analytical results given by (39), (42) & (43), (56) & (57), and (51) & (62), respectively. For example, the approximate WDF in the lower left of figure 8 is available only along the slices where $t = n\Delta/2$, n integer. Along these slices, the aliasing lobes (in frequency) alternate in polarity if n is odd, but remain positive for n even. (Positive lobes are drawn toward the right side in the figure).

Horizontal movement from one diagram to another in figure 8 is accomplished by a Fourier transform from t to υ (or vice versa). Vertical movement is according to a Fourier pair relating τ and f. Finally, the diagonal connection between R_a and A_a, or between W_a and x_a, is by means of a double Fourier transform.
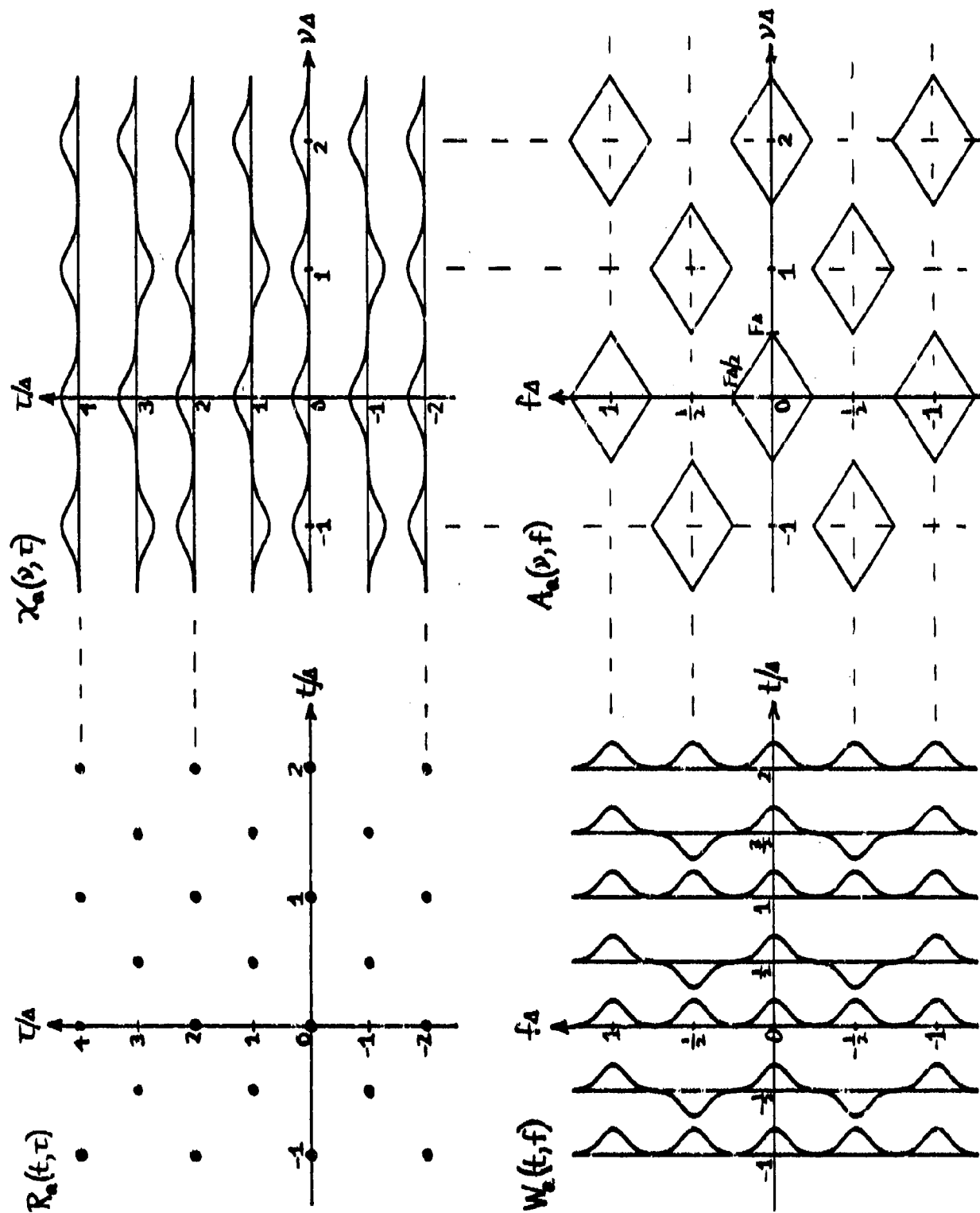
Figure 8. Locations of Available Information

We have seen, by means of (52)-(54) and figure 6, that the original SCF can be recovered from the approximate SCF

$$A_a(\upsilon,f) = \sum_{\substack{q \\ q+m \text{ even}}} \sum_m A(\upsilon - \frac{m}{\Delta}, \; f - \frac{q}{2\Delta}) \qquad \text{for all } \upsilon, f \; , \qquad (63)$$

by means of the diamond gating function $D(\upsilon,f)$ in (53), provided that $\Delta < 1/F$. We have used (51) and (62) here, and dropped the superscripts in accordance with the discussion following (62). This means that we have the possibility of evaluating the original continuous TCF $R(t,\tau)$, WDF $W(t,f)$, and CAF $\chi(\upsilon,\tau)$ at any argument values we please.

SIMPLIFICATION OF SCF $A_a(\upsilon,f)$

It would be an extremely tedious task to evaluate the approximate SCF A_a directly by its definition (46) coupled with (40) and (41), which, in turn, are based upon starting information (39). In fact, there is a startlingly simple way of computing A_a.

Recall from (16) that spectrum

$$\tilde{S}(f) = \Delta \sum_k \exp(-i2\pi f \Delta k) \; s(k\Delta) =$$

$$= \sum_k S(f - \frac{k}{\Delta}) \qquad \text{for all } f \; . \qquad (64)$$

Therefore

$$\tilde{S}(f + \tfrac{\nu}{2})\ \tilde{S}^*(f - \tfrac{\nu}{2}) = \sum_k \sum_{\ell} S(f + \tfrac{\nu}{2} - \tfrac{k}{\Delta})\ S^*(f - \tfrac{\nu}{2} - \tfrac{\ell}{\Delta}) =$$

$$= \sum_k \sum_{\ell} A\left(\nu - \frac{k - \ell}{\Delta}, f - \frac{k + \ell}{2\Delta}\right) \quad \text{for all } \nu, f , \tag{65}$$

where we used (35). Now let $m = k - \ell$ and $q = k + \ell$; then $\frac{m + q}{2} = k$, $\frac{q - m}{2} = \ell$, meaning that $m \pm q$ must always be even. Therefore, (65) can be expressed as

$$\tilde{S}(f + \tfrac{\nu}{2})\ \tilde{S}^*(f - \tfrac{\nu}{2}) = \sum_{q} \sum_{\substack{m \\ q+m \text{ even}}} A\left(\nu - \frac{m}{\Delta}, f - \frac{q}{2\Delta}\right) \quad \text{for all } \nu, f . \tag{66}$$

But (66) is identical with (63)! Thus we have the compact result for the approximate SCF

$$A_a(\nu, f) = \breve{S}(f + \tfrac{\nu}{2})\ \breve{S}^*(f - \tfrac{\nu}{2}) \quad \text{for all } \nu, f , \tag{67}$$

where

$$\tilde{S}(f) = \Delta \sum_k \exp(-i2\pi f \Delta k)\ s(k\Delta) \quad \text{for all } f , \tag{68}$$

in terms of the original time samples $\{s(k\Delta)\}$.

It is convenient at this point to define, for all f, the function

$$\bar{S}(f) = \tilde{S}(f)\ \text{rect}(\Delta f) = \begin{cases} \Delta \sum_k \exp(-i2\pi f \Delta k)\ s(k\Delta) & \text{for } |f| < \frac{1}{2\Delta} \\ \\ 0 & \text{otherwise} \end{cases} , \tag{69}$$

which can be computed directly from the samples $\{s(k\Delta)\}$. Then since $\Delta < 1/F$, reference to figure 2 and (18) reveals that

$$\bar{S}(f) = S(f) \qquad \text{for all } f . \tag{70}$$

The only reason for distinguishing between $\bar{S}$ and S is that we think of $\bar{S}$ as being computed directly from samples $\{s(k\Delta)\}$ via (69), whereas we think of S as being computed from $s(t)$ via Fourier transform (13). Strictly, since $S(f)$ is bandlimited to $\pm F/2$, $\bar{S}(f)$ in (69) only needs to be computed in that somewhat smaller range of f.

At this point, we refer back to (52)-(54) and figure 6 to find that

$$A(\nu,f) = A_a(\nu,f) \ D(\nu,f) = \bar{S}(f + \tfrac{\nu}{2}) \ \bar{S}^*(f - \tfrac{\nu}{2}) \qquad \text{for all } \nu,f , \tag{71}$$

since only the origin lobe in figure 6 can contribute, and there is no overlap. Thus we have a very direct way of recovering the original SCF from the time data samples: compute $\bar{S}(f)$ from (69), and then $A(\nu,f)$ from (71). All these results are predicated on sampling rate condition $\Delta < 1/F$; they do not require $\Delta < (2F)^{-1}$.

If we substitute (70) in (71), we have original definition (35). Thus we have come full circle on the SCF, returning with an obvious relation. This indicates that a shortcut could have been taken with regard to getting the key result (67). We have pursued the longer route because it indicates what the complete set of fundamental processing equations are, and it clarifies a number of points that have been under contention in the literature.

RECOVERY OF ORIGINAL WDF AND CAF

From (37) and (71), we obtain the original continuous WDF as

$$W(t,f) = \int d\upsilon \, \exp(i2\pi\upsilon t) \, \bar{S}(f + \tfrac{\upsilon}{2}) \, \bar{S}^*(f - \tfrac{\upsilon}{2}) \qquad \text{for all } t,f \, , \qquad (72)$$

where $\bar{S}(f)$ is given by (69) in terms of samples $\{s(k\Delta)\}$. The truncation of $\bar{S}(f)$ at $f = \pm (2\Delta)^{-1}$ in (69) is what prevents all the distant sidelobes of $A_a(\upsilon,f)$ from contributing. We could hardly have expected a simpler result.

From (37) and (71), we also obtain the original CAF according to

$$\chi(\upsilon,\tau) = \int df \, \exp(i2\pi f\tau) \, \bar{S}(f + \tfrac{\upsilon}{2}) \, \bar{S}^*(f - \tfrac{\upsilon}{2}) \qquad \text{for all } \upsilon,\tau \, . \qquad (73)$$

Thus, both the WDF and the CAF can be recovered by single Fourier transforms of the same product function, but on complementary variables υ and f, respectively.

RECOVERY OF ORIGINAL TCF

Probably the best way to recover the original TCF $R(t,\tau)$ is by means of a combination of (32), (5), and (70):

$$R(t,\tau) = s(t + \tfrac{\tau}{2}) \, s^*(t - \tfrac{\tau}{2}) \, , \qquad (74)$$

with

$$s(t) = \int df \, \exp(i2\pi ft) \, \bar{S}(f) \, . \qquad (75)$$

All the above procedures employ $\bar{S}(f)$ and a Fourier transform in some fashion. The quantity $\bar{S}(f)$ can be computed at any f of interest, directly from samples $\{s(k\Delta)\}$, by means of (69).

DIRECT TIME DOMAIN RECOVERY OF CONTINUOUS WDF

We have given two alternatives for the recovery of continuous time waveform s(t) from samples $\{s(k\Delta)\}$. They are (21) or (75) & (69). If we employ the former in the time definition of the original WDF (line 1 of (26)), we find, for all t,f,

$$W(t,f) = \int d\tau \, \exp(-i2\pi f\tau) \, s(t + \tfrac{\tau}{2}) \, s^*(t - \tfrac{\tau}{2}) =$$

$$= \Delta \sum_k \sum_\ell \exp[-i2\pi f\Delta(k - \ell)] \, s(k\Delta) \, s^*(\ell\Delta) \, W_0\left(t - \frac{k + \ell}{2}\Delta, f\right) , \qquad (76)$$

where

$$W_0(t,f) = \begin{cases} \dfrac{\sin[2\pi(1 - 2|f|\Delta) \, t/\Delta]}{\pi t/\Delta} & \text{for } |f| < \dfrac{1}{2\Delta} \\[2ex] 0 & \text{for } |f| > \dfrac{1}{2\Delta} \end{cases} \qquad \text{for all } t . \qquad (77)$$

This result is equivalent to [5, (5)&(6)]. However, as noted there, this alternative for the WDF is not computationally attractive, although (76) is certainly alias-free because it is restoring W(t,f) itself, and not some approximation to it.

As an aside, if the frequency domain version of the WDF is used instead (line 2 of (26)), and if (21) is immediately transformed into the frequency domain, we get directly

$$S(f) = \tilde{S}(f) \, \text{rect}(\Delta f) = \bar{S}(f) , \tag{78}$$

in complete agreement with (72).

In the sequel to (62), it was mentioned that an alternative approach involving impulsive sampling could be used to get various impulsive two-dimensional functions from samples $\{s(k\Delta)\}$. In a similar vein, the continuous two-dimensional functions can be recovered by direct convolution (interpolation) in the domain(s) of interest. These alternative forms are not as numerically useful as the ones presented above, and so are deferred to appendix C. However, some useful insight into the inadequacy of some past attempts at interpolation is gained by this alternative viewpoint, and the readers attention is directed to these results.

DISCUSSION

In retrospect, (72) and (69) are an obvious result. We know that the original continuous WDF is given by (line 2 of (26))

$$W(t,f) = \int d\nu \, \exp(i2\pi\nu t) \, S(f + \tfrac{\nu}{2}) \, S^*(f - \tfrac{\nu}{2}) \qquad \text{for all } t,f . \tag{79}$$

So if we can get S(f) exactly from samples $\{s(k\Delta)\}$, in some (any) fashion, we can get W via (79). But, in fact, $\tilde{S}(f)$ in (69) is identically equal to S(f) for all f, when $\Delta < 1/F$. Condition $\Delta < (2F)^{-1}$ is patently unnecessary and too restrictive. A similar comment holds with regard to the CAF.

Given samples $\{s(k\Delta)\}$, the function $\bar{S}(f)$ in (69) can be computed at any desired f values of interest. Therefore the product

$$\bar{S}(f + \tfrac{\nu}{2}) \; \bar{S}^*(f - \tfrac{\nu}{2}) \tag{80}$$

required in (72) and (73) can be computed at any ν,f values needed, and the integrals for $W(t,f)$ and $\chi(\nu,\tau)$ evaluated very accurately at any arguments of interest.

This is the major difference relative to the TCF $R(t,\tau)$, (74), which could only be calculated at interspersed points in the t,τ plane from the available data; see (39) and figure 4. Strictly, waveform $s(t)$ could be interpolated, and then TCF $R(t,\tau)$ filled in at the intermediate points of interest in figure 4. This viable alternative requires just slightly more calculations that the frequency domain approach given above; we will discuss and compare both alternatives in a later section.

In practice, $\bar{S}(f)$ will only be calculated at a discrete set of frequencies, in order to economize on computational effort. We then find that the product function (80) is available at interspersed points in the ν,f plane in an identical manner to that for TCF $R(t,\tau)$ in figure 4. In fact, if $\bar{S}(f)$ is computed only for f/Δ_f = integer, then (80) is available only at

$$f/\Delta_f = \text{integer} , \qquad \nu/\Delta_f = \text{even integer} \tag{81}$$

and at

$$f/\Delta_f = \text{odd integer}/2 , \qquad \nu/\Delta_f = \text{odd integer} . \tag{82}$$

Just as this type of interspersed sampling required a finer sampling interval in the time domain (see figure 5 and (45)), so also is a finer increment required here in the frequency domain. Namely, we must have

$$\Delta_f < \frac{1}{2T} , \qquad (83)$$

in order to avoid aliasing in the t domain of the reconstructed WDF via Fourier transform (72). The same requirement holds for aliasing control in the τ domain of the reconstructed CAF via (73). Here, T is the overall effective duration of waveform s(t):

$$\left| s(t) \right| \approx 0 \quad \text{for } |t| > T/2 . \qquad (84)$$

See figure 9. (The waveform can be centered at t = 0 without loss of generality, merely by time delaying it.) However, there is a very convenient and efficient way to meet requirement (83), as will be shown shortly, whereas requirement (45), $\Delta < (2F)^{-1}$, is very unattractive, at least through direct sampling of time domain waveform s(t).

Since the total extent of the spectrum S(f) is F Hz (see (14) and figure 1), waveform s(t) cannot be strictly time-limited. However, we assume that a finite T value can be found for figure 9 such that approximation (84) is a good one. Strictly, (84) should read

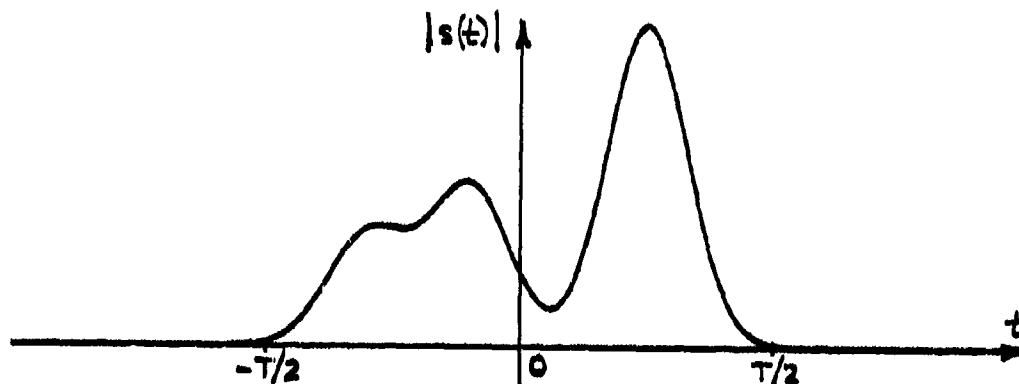$$\frac{|s(t)|}{\max |s(t)|} \ll 1 \quad \text{for } |t| > T/2 . \qquad (85)$$

Figure 9. Waveform s(t)

DISCRETE PROCESSING IN FREQUENCY DOMAIN

Thus far, we have not discretized in the frequency domain; both ν and f have been allowed to take on continuous values. So we have, from (69),

$$\bar{S}(f) = \Delta \sum_k \exp(-i2\pi f \Delta k) \, s(k\Delta) \qquad \text{for } |f| < \frac{1}{2\Delta} . \tag{86}$$

Now, suppose that we only evaluate $\bar{S}(f)$ at a set of discrete frequencies, according to

$$\bar{S}(\frac{n}{N\Delta}) = \Delta \sum_k \exp(-i2\pi nk/N) \, s(k\Delta) \qquad \text{for } |n| < \frac{N}{2} . \tag{87}$$

Since the sum on the right-hand side of (87) has period N in n, we can evaluate it quickly via a collapsed N-point FFT; see (10)-(12). The negative n values desired in (87) are easily accommodated by means of a modulo N look-up in the FFT output.

EVALUATION OF WDF

The increment in argument f of $\bar{S}(f)$ in (87) is

$$\Delta_f = \frac{1}{N\Delta} . \tag{88}$$

In order to use these results in approximating integral (72) for the WDF,

$$W(t,f) = \int d\nu \, \exp(i2\pi \nu t) \, \bar{S}(f + \frac{\nu}{2}) \, \bar{S}^*(f - \frac{\nu}{2}) , \tag{89}$$

we need to have the increment in ν satisfy (due to the $\nu/2$ arguments)

$$\frac{1}{2} \Delta_v = \frac{1}{N\Delta} , \qquad \text{i.e.,} \qquad \Delta_v = \frac{2}{N\Delta} . \tag{90}$$

So if we limit v in (89) to the values $\frac{2\ell}{N\Delta}$, one possible Trapezoidal approximation to W is

$$\tilde{W}(t,f) \equiv \frac{2}{N\Delta} \sum_{\ell} \exp(i2\pi \frac{2\ell}{N\Delta} t) \; \tilde{S}(f + \frac{\ell}{N\Delta}) \; \tilde{S}^*(f - \frac{\ell}{N\Delta}) \qquad \text{for all } t,f . \tag{91}$$

This function is defined on a continuum in time,frequency space. But this can be developed according to

$$\tilde{W}(t,f) = \int dv \, \exp(i2\pi vt) \; \tilde{S}(f + \frac{v}{2}) \; \tilde{S}^*(f - \frac{v}{2}) \frac{2}{N\Delta} \sum_{\ell} \delta(v - \frac{2\ell}{N\Delta}) =$$

$$= W(t,f) \overset{t}{\circledast} \sum_{\ell} \delta(t - \ell \frac{N\Delta}{2}) =$$

$$= \sum_{\ell} W(t - \ell \frac{N\Delta}{2}, f) \qquad \text{for all } t,f , \tag{92}$$

where we used (72).

Since waveform $s(t)$ is approximately limited to $|t| < T/2$ (see (84) and figure 9), then WDF $W(t,f)$ is also approximately limited to $|t| < T/2$, as may be seen from line 1 of (26). The approximation (92) then appears as in figure 10.
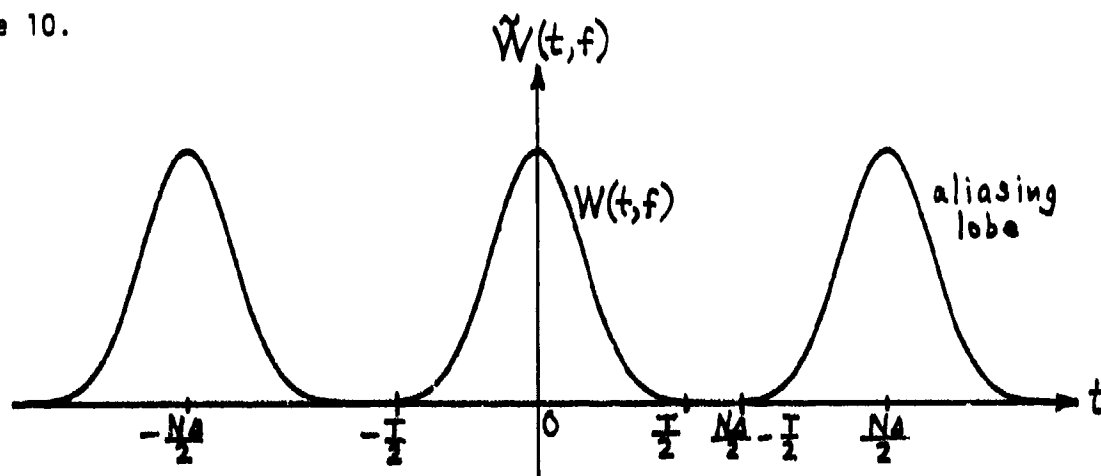

Figure 10. Time-Aliased WDF

In order that approximate WDF $\tilde{W}$ not suffer significant overlap in time, we see that we need

$$\frac{T}{2} < \frac{N\Delta}{2} - \frac{T}{2} , \qquad \text{i.e., } N > \frac{2T}{\Delta} . \tag{93}$$

The FFT size, N, in (87) must be at least twice as large as the number of samples, T/Δ, taken of waveform $s(t)$. Recalling (88), this inequality becomes

$$\Delta_f = \frac{1}{N\Delta} < \frac{1}{2T} , \tag{94}$$

consistent with (83), as predicted. Thus, approximation $\tilde{W}(t,f)$ is an extremely good approximation to $W(t,f)$ for $|t| < T/2$ if FFT size N satisfies (93). The goodness of $\tilde{W}$ depends critically on the degree of satisfaction of (85).

More generally, we could limit the ν values in (89) to

$$\nu = \nu_0 + \frac{2\ell}{N\Delta} , \qquad \nu_0 \text{ arbitrary} \qquad (\Delta_\nu = \frac{2}{N\Delta}) , \tag{95}$$

getting alternative approximation

$$\tilde{W}_a(t,f) \equiv \int d\nu \, \exp(i2\pi t\nu) \, \tilde{S}(f + \frac{\nu}{2}) \, \tilde{S}^*(f - \frac{\nu}{2}) \, \frac{2}{N\Delta} \sum_\ell \delta(\nu - \nu_0 - \frac{2\ell}{N\Delta}) =$$

$$= W(t,f) \, \overset{t}{\circledast} \left\{ \exp(i2\pi t\nu_0) \sum_\ell \delta\left(t - \ell \, \frac{N\Delta}{2}\right) \right\} =$$

$$= \sum_\ell W(t - \ell \, \frac{N\Delta}{2}, f) \, \exp(i\pi\nu_0 N\Delta\ell) \qquad \text{for all } t,f . \tag{96}$$

The plot for $\left|\tilde{W}_a(t,f)\right|$ is identical to that of $\left|\tilde{W}(t,f)\right|$ in figure 10, since the magnitude of phase factor $\exp(i\pi\nu_0 N\Delta\ell)$ is 1 for all ℓ. The main lobe, $\ell = 0$, is unaffected by the choice of ν_0. So criterion (93) is again sufficient to avoid time-aliasing in $\tilde{W}_a$, regardless of shift ν_0.

DISCRETIZATION IN TIME AND FREQUENCY

For convenience, we therefore return to $\tilde{W}$ in (91) and get, in particular, the values

$$\tilde{W}(t,\frac{n}{N\Delta}) = \frac{2}{N\Delta} \sum_{\ell} \exp(j2\pi \frac{2\ell}{N\Delta} t) \bar{S}(\frac{n+\ell}{N\Delta}) \bar{S}^*(\frac{n-\ell}{N\Delta}) \quad \text{for all } t , \quad (97)$$

where we must choose frequency $f = \frac{n}{N\Delta}$, in order to use the available samples of $\bar{S}$ in (87). We now further choose time

$$t = \frac{m\Delta}{2} \frac{N}{M} \tag{98}$$

and get the approximation in the form

$$\tilde{W}(\frac{m\Delta}{2} \frac{N}{M},\frac{n}{N\Delta}) = \frac{2}{N\Delta} \sum_{\ell} \exp(j2\pi m\ell/M) \bar{S}(\frac{n+\ell}{N\Delta}) \bar{S}^*(\frac{n-\ell}{N\Delta}) . \tag{99}$$

The reason for this choice of t values is that this sum can be accomplished as a collapsed M-point FFT as described in (10)-(12). The range of values that must be covered is

$$\frac{|m|\Delta}{2} \frac{N}{M} < \frac{T}{2} , \quad \text{i.e., } |m| < \frac{T}{\Delta} \frac{M}{N} \tag{100}$$

and

$$\frac{|n|}{N\Delta} < \frac{F}{2} , \quad \text{i.e., } |n| < F\Delta \frac{N}{2} . \tag{101}$$

Coincidentally, this identical procedure above has already been derived by the author in [9, (A-13)&(A-14)]. However, it was done, at that time, to generate slices in time of the WDF, without realizing that the procedure also had an alias-elimination feature. Requirement (93) was [9, (A-5)&(A-6)].

INCREMENTS IN t AND f

The increments in t and f in approximation $\widetilde{W}(t,f)$ in (99) are

$$\Delta_t = \frac{\Delta}{2}\frac{N}{M} \quad \text{and} \quad \Delta_f = \frac{1}{N\Delta} . \tag{102}$$

Since the original WDF is given by

$$W(t,f) = \int d\tau \, \exp(-i2\pi f\tau) \, R(t,\tau) , \tag{103}$$

and the effective extent of $R(t,\tau)$ in τ is $\pm T$ for the waveform $s(t)$ satisfying (84), then we must require

$$\Delta_f < \frac{1}{2T} , \quad \text{i.e., } N > \frac{2T}{\Delta} , \tag{104}$$

in order to track the variation of $W(t,f)$ in f. (See appendix A.) However, this was a condition already encountered in (93).

Furthermore, since we have the alternative Fourier transform

$$W(t,f) = \int d\upsilon \, \exp(i2\pi\upsilon t) \, A(\upsilon,f) , \tag{105}$$

and the extent of $A(\upsilon,f)$ in υ is $\pm F$, then we must also have

$$\Delta_t < \frac{1}{2F} , \quad \text{i.e., } \Delta < \frac{1}{F}\frac{M}{N} , \tag{106}$$

in order to track $W(t,f)$ variations in t. Now if we choose M smaller than N, say $M = N/2$, then we obtain condition $\Delta < (2F)^{-1}$. But this is a finer time sampling increment than required. Also (102) gives $\Delta_t = \Delta$, which does not track $W(t,f)$ adequately in time; see (24)-(31). Conversely, if we choose M larger than N, say $M = 2N$, then we get $\Delta < 2/F$, which is already accommodated by the earlier requirement

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$\Delta < 1/F$. And $\Delta_t = \Delta/4$, which is overly fine in time. So, in order to minimize the range of m values needed for investigation, we choose

$$M = N . \tag{107}$$

Notice that the time increment Δ_t in (102) for $\tilde{W}(t,f)$ is then $\Delta_t = \Delta/2$, not the Δ that was sufficient for sampling $s(t)$. This is consistent with the fact that $W(t,f)$ can be sharper in t than $s(t)$.

SUMMARY OF WDF EQUATIONS

Here we summarize the major assumptions and requirements and list the major equations by which the approximate WDF $\tilde{W}$ can be computed.

Assumptions:
$$S(f) = 0 \quad \text{for } |f| > F/2 ,$$
$$|s(t)| \approx 0 \quad \text{for } |t| > T/2 . \tag{108}$$

This is no loss of generality, since the waveform can be time delayed and frequency shifted as desired.

Requirements:
$$\Delta < \frac{1}{F} ,$$
$$N > \frac{2T}{\Delta} \quad (> 2TF) . \tag{109}$$

Equations:
$$S\left(\frac{n}{\Delta}\right) = \begin{cases} \Delta \sum_k \exp(-12\pi nk/N) \, s(k\Delta) & \text{for } |n| < \frac{N}{2} \\ 0 & \text{otherwise} \end{cases} . \tag{110}$$

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$$\widetilde{W}(\frac{m\Delta}{2},\frac{n}{N\Delta}) = \frac{2}{N\Delta} \sum_{\ell} \exp(i2\pi m\ell/N) \ \bar{S}(\frac{n+\ell}{N\Delta}) \ \bar{S}^{*}(\frac{n-\ell}{N\Delta})$$

$$\text{for } |m| < \frac{T}{\Delta}, \quad |n| < \frac{N}{2} F\Delta. \tag{111}$$

Operation (110) can be accomplished by a single N-point FFT with collapsing, while (111) requires an N-point FFT for each n value of interest. The latter N-point FFT (for a given n) will sweep out N values of integer m; this will cover a total time range of $N \frac{\Delta}{2} > T$, as desired. The total number of n (frequency) values to be searched is $NF\Delta$. Values of $\widetilde{W}$ for negative values of m are available in location m modulo N of the FFT output.

For the most advantageous choices of

$$\Delta = \frac{1}{F}, \quad N = \frac{2T}{\Delta} = 2TF, \tag{112}$$

we have ranges

$$|m| < TF, \quad |n| < TF.$$

EVALUATION OF CAF

The equations for calculation of the CAF, from discretized frequency samples of $\bar{S}(f)$, are very similar to those given above for the WDF. Accordingly, they are deferred to appendix D.

INTERPOLATION OF TIME WAVEFORM

It was mentioned in the sequel to (80) that available data samples $\{s(k\Delta)\}$ could be interpolated in time, to fill in the vacant spots of TCF $R(t,\tau)$ in figure 4. One procedure to accomplish this is by direct use of sinc-function interpolation in (21). However, a more efficient procedure is by use of FFTs.

From (22), we can get samples of the approximate spectrum according to

$$\tilde{S}(\frac{m}{M\Delta}) = \Delta \sum_k \exp(-i2\pi mk/M) \, s(k\Delta) \qquad \text{for } |m| \leq \frac{M}{2} , \tag{113}$$

which can be accomplished by an M-point FFT. The frequency increment is $\Delta_f = (M\Delta)^{-1}$. The original continuous time waveform is

$$s(t) = \int df \, \exp(i2\pi ft) \, S(f) =$$

$$= \int_{-(2\Delta)^{-1}}^{(2\Delta)^{-1}} df \, \exp(i2\pi ft) \, \tilde{S}(f) =$$

$$\tilde{=} \frac{1}{M\Delta} \sum_{m=-M/2}^{M/2} \exp(i2\pi \frac{m}{M\Delta} t) \, \tilde{S}(\frac{m}{M\Delta}) =$$

$$\equiv \hat{s}(t) \qquad \text{for all } t , \tag{114}$$

where we have utilized the fact that $\Delta < 1/F$. (The tic mark on the summation symbol indicates that the summand values at $m = \pm M/2$ must be scaled by 1/2.) But approximation $\hat{s}(t)$ can be developed in the form

$$\hat{s}(t) = \int_{\frac{-1}{2\Delta}}^{\frac{1}{2\Delta}} df \exp(i2\pi ft)\, \tilde{S}(f)\, \frac{1}{M\Delta} \sum_{m=-M/2}^{M/2\,\prime} \delta(f - \frac{m}{M\Delta}) =$$

$$= \int df \exp(i2\pi ft)\, S(f)\, \frac{1}{M\Delta} \sum_{m} \delta(f - \frac{m}{M\Delta}) =$$

$$s(t) \circledast \sum_{m} \delta(t - mM\Delta) = \sum_{m} s(t - mM\Delta) \quad \text{for all } t. \qquad (115)$$
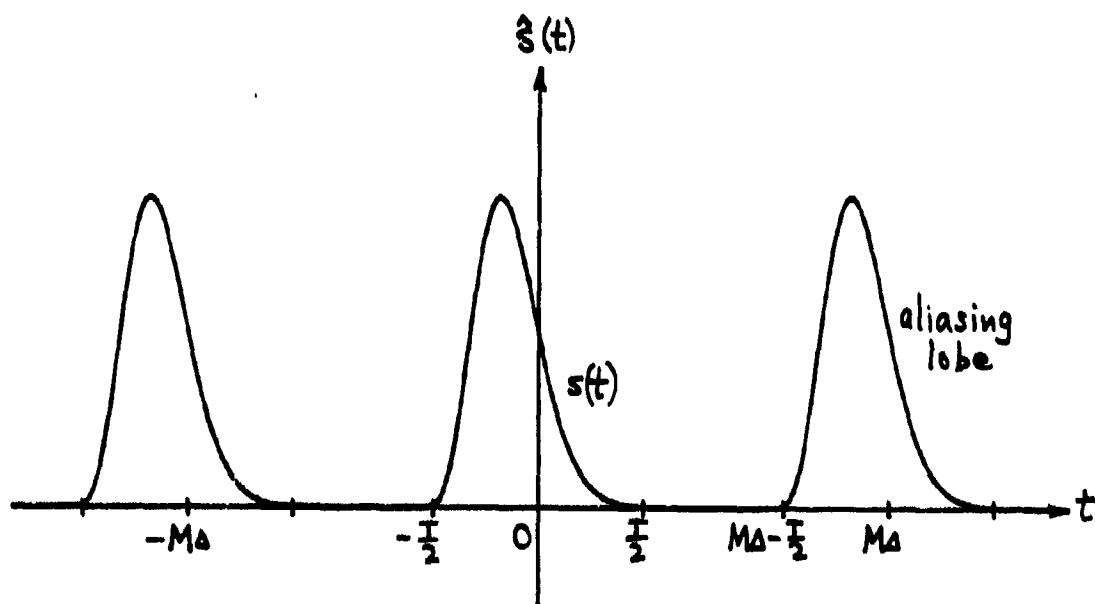
This aliased function is illustrated in figure 11.

Figure 11. Time-Aliased Waveform $\hat{s}(t)$

In order to avoid time overlap in figure 11, we must have

$$M > \frac{T}{\Delta} \qquad (> TF) . \tag{116}$$

If this FFT size requirement for (113) is met, then figure 11 reveals that

$$\hat{s}(t) \cong s(t) \qquad \text{for } |t| < \frac{M\Delta}{2} ; \tag{117}$$

that is, we can expect that $\hat{s}(t)$ is a good approximation to $s(t)$.

In particular, if we want to interpolate samples $\{s(k\Delta)\}$ by a factor of 2, we have, from (117) and (114),

$$s(\frac{n\Delta}{2}) \cong \hat{s}(\frac{n\Delta}{2}) = \frac{1}{M\Delta} \sum_{m=-M/2}^{M/2} \exp(j2\pi \frac{mn}{2M}) \tilde{S}(\frac{m}{M\Delta}) \qquad \text{for } |n| < M . \tag{118}$$

This can be done by a 2M-point FFT with zero-fill, and $\left\{ \tilde{S}(\frac{m}{M\Delta}) \right\}_{-M/2}^{M/2}$ are made available by the M-point FFT in (113). The two requirements that must be met are

$$\Delta < \frac{1}{F}, \qquad 2M > \frac{2T}{\Delta} \qquad (> 2TF) . \tag{119}$$

This is the procedure utilized in [8].

DISCUSSION/SUMMARY

We have two alternatives for computing an alias-free WDF and CAF. The requirements that must be met are

$$\Delta < \frac{1}{F}, \qquad M > \frac{T}{\Delta}, \qquad N = 2M > \frac{2T}{\Delta}. \tag{120}$$

The processing equations are summarized below.

TIME DOMAIN APPROACH

$$\overline{S}(\frac{m}{M\Delta}) = \Delta \sum_k \exp(-i2\pi mk/M) \, s(k\Delta) \qquad \text{for } |m| \leq \frac{M}{2}, \tag{121}$$

$$\hat{s}(\frac{n\Delta}{2}) = \begin{cases} \dfrac{1}{M\Delta} \displaystyle\sum_{m=-M/2}^{M/2} \exp(i2\pi mn/N) \, \overline{S}(\frac{m}{M\Delta}) & \text{for } |n| < M \\[2em] 0 & \text{otherwise} \end{cases}, \tag{122}$$

$$\hat{W}(\frac{m\Delta}{2}, \frac{n}{N\Delta}) = \Delta \sum_{\ell} \exp(-i2\pi n\ell/N) \, \hat{s}(\frac{m+\ell}{2}\Delta) \, \hat{s}^*(\frac{m-\ell}{2}\Delta)$$

$$\text{for } |m| < \frac{T}{\Delta}, \qquad |n| < \frac{N}{2} F\Delta < \frac{N}{2}. \tag{123}$$

In the last equation, we obtain slices of the WDF in frequency (n) at fixed time (m). (This result is in essential agreement with [7, (77) and (74)]; however, we are not restricted here to T = MΔ.)

FREQUENCY DOMAIN APPROACH

$$\bar{S}(\frac{n}{N\Delta}) = \begin{cases} \Delta \sum_k \exp(-i2\pi nk/N)\, s(k\Delta) & \text{for } |n| < \frac{N}{2}, \\ \\ 0 \text{ otherwise} \end{cases} \qquad (124)$$

$$\tilde{W}(\frac{m\Delta}{2}, \frac{n}{N\Delta}) = \frac{2}{N\Delta} \sum_{\ell} \exp(i2\pi m\ell/N)\, \bar{S}(\frac{n+\ell}{N\Delta})\, \bar{S}^*(\frac{n-\ell}{N\Delta})$$

$$\text{for } |m| < \frac{T}{\Delta} < \frac{N}{2}, \qquad |n| < \frac{N}{2} F\Delta. \qquad (125)$$

Here, we obtain slices of the WDF in time (m) at fixed frequency (n). (This result is in essential agreement with [7, (81)]; however, we are not restricted here to $2T = N\Delta$.)

COMPARISON

Since waveform $s(t) \cong 0$ for $|t| > T/2$, the sum on k in (121) can be confined to $|k| < \frac{T}{2\Delta} < \frac{M}{2} = \frac{N}{4}$, while the sum on k in (124) can similarly be confined to $|k| < \frac{T}{2\Delta} < \frac{N}{4}$.

The time domain approach requires one M-point (M = N/2) FFT, one N-point FFT, and then an N-point FFT for each time index m of interest. The frequency domain approach requires one N-point FFT and then an N-point FFT for each frequency index n of interest. Thus, the time domain approach requires one additional $M = \frac{N}{2}$ – point FFT, which is a negligible difference, compared with the multiple FFTs that must be performed for (123) or (125). The major difference between the two approaches appears to be whether one wants slices in frequency, or slices in time, of the WDF.

It should be emphasized that the procedure here reconstructs the original WDF of the continuous waveform s(t), at any time,frequency (t,f) arguments of interest, from the samples $\{s(k\Delta)\}$. There is no need to define or set up some arbitrary discrete form of the WDF. The discretization of the t,f arguments of the WDFs is undertaken only **after** this reconstruction procedure has been delineated (via the υ,f plane) and the sufficiency of sampling requirement $\Delta < 1/F$ established. Of course, this eventual discretization in time/frequency is necessary in order to reduce the general procedure to a practical efficient algorithm. A similar set of arguments applies equally well to the reconstruction of the CAF.

An alternative philosophy, to developing Trapezoidal approximations for the various two-dimensional functions, is given in appendix B in terms of a pair of interspersed impulsive trains in the t,τ plane. The end result for the approximate SCF in the υ,f plane is shown to be identical to that obtained earlier.

The connection of this two-dimensional impulse train with direct time domain sampling of the waveform s(t) is considered in appendix E. Again, the two approaches, time domain sampling versus t,τ domain sampling, are shown to yield the same result. Finally, the fundamental rules and patterns relating two-dimensional interspersed infinite sampling trains are displayed at the end of appendix E.

APPENDIX A. EXTENTS AND RATES OF VARIATION
OF TCF, WDF, CAF, SCF

From (5), we have Fourier pair

$$S(f) = \int dt \, \exp(-i2\pi ft) \, s(t) \, ,$$

$$s(t) = \int df \, \exp(i2\pi ft) \, S(f) \, ; \qquad (A-1)$$

and from figures 1 and 2, we know that if

$$S(f) = 0 \quad \text{for } |f| > F/2 \, , \qquad (A-2)$$

then samples $\{s(k\Delta_t)\}$ are sufficient for reconstruction of s(t) if $\Delta_t < 1/F$. That is, if a spectrum is bandlimited in the frequency domain to total extent F Hz, samples of the corresponding time function must be taken with time increment $\Delta_t < 1/F$, in order not to lose any significant information.

In a similar vein, the duality of the equations in (A-1) indicates that if, instead,

$$s(t) = 0 \quad \text{for } |t| > T/2 \, , \qquad (A-3)$$

then spectrum samples $\{S(n\Delta_f)\}$ are sufficient for reconstruction of S(f) if frequency increment $\Delta_f < 1/T$.

The general rule, here, is that if a function in one domain is essentially limited to overall extent E, samples in its Fourier transform domain must be taken finer than 1/E, in order not to lose any information. This rule will be used frequently below.

EXTENTS OF TCF, WDF, CAF, AND SCF

Henceforth, we assume that

$$S(f) \approx 0 \quad \text{for } |f| > F/2$$

and

$$s(t) \approx 0 \quad \text{for } |t| > T/2 . \tag{A-4}$$

Thus, the overall frequency and time extents are approximately F Hz and T seconds, respectively. It then readily follows from (32)-(38), namely,

$$R(t,\tau) = s(t + \tfrac{\tau}{2}) \, s^*(t - \tfrac{\tau}{2}) , \tag{A-5}$$

$$A(\nu,f) = S(f + \tfrac{\nu}{2}) \, S^*(f - \tfrac{\nu}{2}) , \tag{A-6}$$

$$W(t,f) = \int d\tau \, \exp(-i2\pi f\tau) \, R(t,\tau) = \tag{A-7}$$

$$= \int d\nu \, \exp(i2\pi \nu t) \, A(\nu,f) , \tag{A-8}$$

$$\chi(\nu,\tau) = \int dt \, \exp(-i2\pi \nu t) \, R(t,\tau) = \tag{A-9}$$

$$= \int df \, \exp(i2\pi f\tau) \, A(\nu,f) , \tag{A-10}$$

that the extents of these functions are as depicted in figure A-1. The solid curves depict the contour level within which the function is essentially concentrated. In fact, for Gaussian waveform $s(t) = a \exp(-\frac{t^2}{2\sigma^2})$, the choices $T = 4\sigma$, $F = \frac{2}{\pi\sigma}$, for example, give these exact results in figure A-1, at the $\exp(-4) = .018$ level.
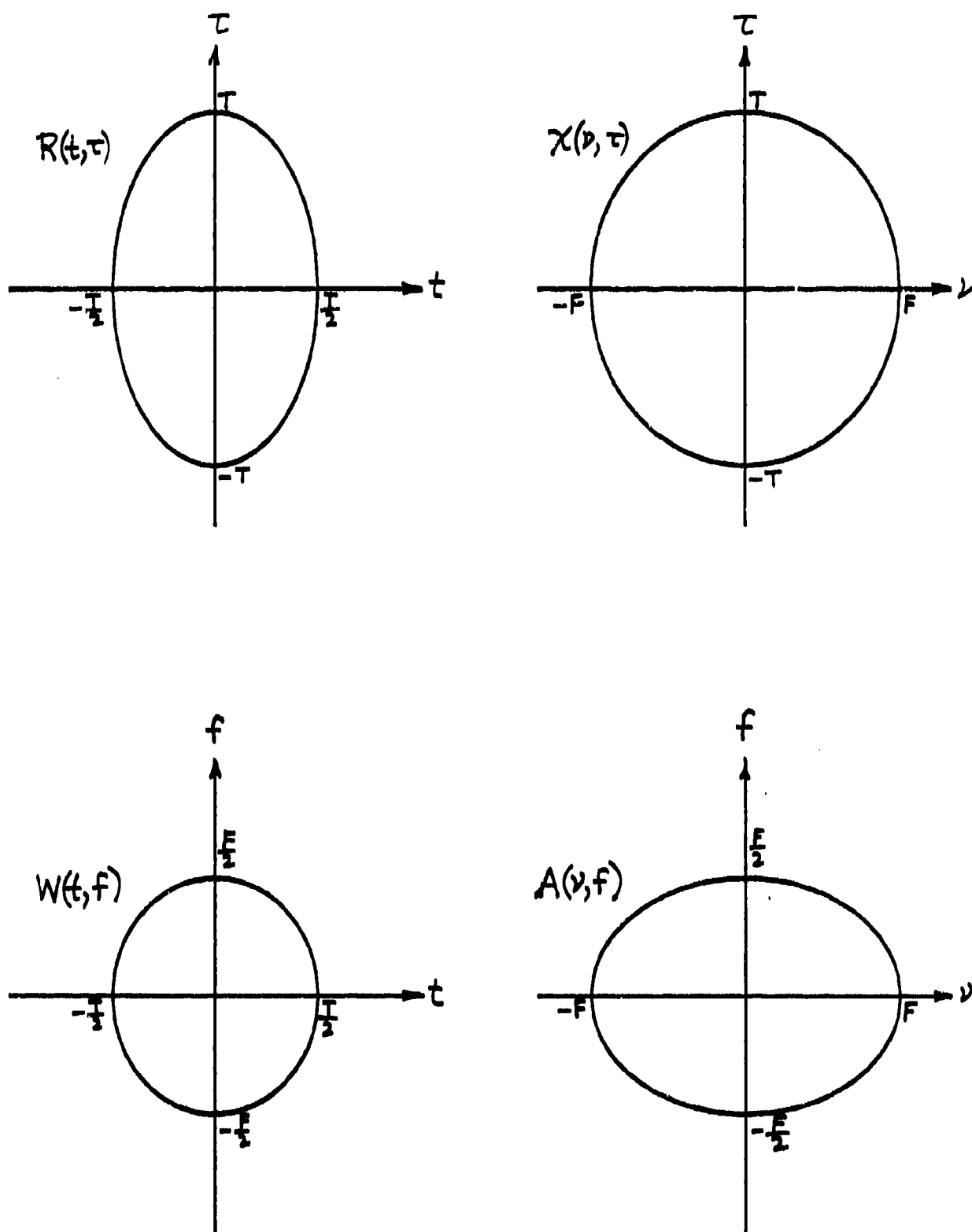
Figure A-1. Extents of the Two-Dimensional Functions

RATES OF VARIATION

We now combine the sampling rule deduced under (A-3) with the extents in figure A-1. From (A-7) and (A-8), we must have sampling increments (i.e., spacings of argument values on left-hand side) satisfy

$$\left. \begin{array}{l} \Delta_f < \frac{1}{2T} \\[2em] \Delta_t < \frac{1}{2F} \end{array} \right\} \quad \text{to adequately track WDF .} \tag{A-11}$$

From (A-9) and (A-10), we need

$$\left. \begin{array}{l} \Delta_\upsilon < \frac{1}{T} \\[2em] \Delta_\tau < \frac{1}{F} \end{array} \right\} \quad \text{to track CAF .} \tag{A-12}$$

By inverting (A-7)-(A-10), we express

$$R(t,\tau) = \int df \ \exp(i2\pi f\tau) \ W(t,f) = \tag{A-13}$$

$$= \int d\upsilon \ \exp(i2\pi\upsilon t) \ \chi(\upsilon,\tau) \ , \tag{A-14}$$

$$A(\upsilon,f) = \int dt \ \exp(-i2\pi\upsilon t) \ W(t,f) = \tag{A-15}$$

$$= \int d\tau \ \exp(-i2\pi f\tau) \ \chi(\upsilon,\tau) \ . \tag{A-16}$$

There follows from (A-13) and (A-14), that we need

$$\left. \begin{array}{l} \Delta_\tau < \frac{1}{F} \\[2em] \Delta_t < \frac{1}{2F} \end{array} \right\} \quad \text{to track TCF ,} \tag{A-17}$$

while from (A-15) and (A-16),

$$\left.\begin{array}{c} \Delta_v < \dfrac{1}{T} \\[2em] \Delta_f < \dfrac{1}{2T} \end{array}\right\} \text{ to track SCF .} \qquad\qquad \text{(A-18)}$$

These last four restrictions are identical to those given in (A-11) and (A-12). The total number of samples required to completely describe any one of the four two-dimensional characterizations is T^2F^2.

SAMPLING RATE FOR REAL WAVEFORM

Requirement (17) on the time sampling increment, $\Delta < 1/F$, for recovery of the time waveform $s(t)$, is based upon figures 1 and 2 for a complex envelope waveform $s(t)$. If $s(t)$ were, instead, a real waveform $s_1(t)$, the earlier development covers this case as well, but with a change in notation. The spectrum $S_1(f)$ of waveform $s_1(t)$ is symmetric about $f = 0$, as depicted in figure A-2. F is now the total frequency extent of the positive-frequency components of $s_1(t)$.

We now have frequency limit

$$\frac{F_1}{2} = f_c + \frac{F}{2} , \qquad\qquad \text{(A-19)}$$

and the stringent requirement (45), for an unaliased WDF, becomes

$$\Delta < \frac{1}{2F_1} = \frac{1}{4f_c + 2F} . \qquad\qquad \text{(A-20)}$$

For a narrowband waveform, $f_c \gg F$, this requires an unnecessarily high sampling rate, compared to what would be required for the waveform

corresponding to single-sided bandwidth F. Extraction of the complex

envelope (or analytic function) of $s_1(t)$ would return us to WDF requirement

$\Delta < \frac{1}{2F}$, as in (45). This pre-processing feature is recommended for all real

waveforms. However, we also want to avoid this more stringent WDF

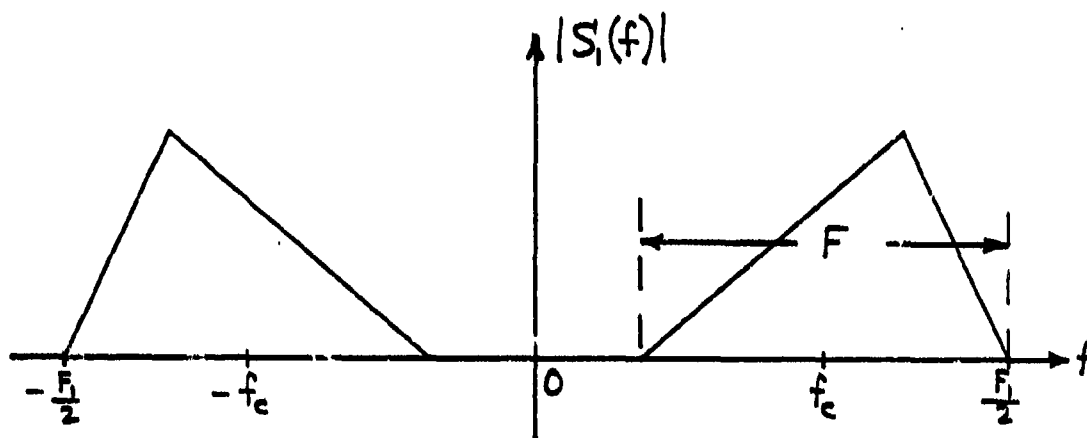requirement and be subject only to the $\Delta < 1/F$ limitation.

Figure A-2. Spectrum of Real Waveform

APPENDIX B.   IMPULSIVE SAMPLING APPROACH

Instead of trying to develop Trapezoidal approximations to the WDF, CAF, and SCF integrals from the available information about the TCF in (39), we adopt here the philosophy that continuous TCF $R(t,\tau)$ has had a pair of impulsive trains applied to it, yielding the impulsive approximation

$$R_1(t,\tau) \equiv \Delta^2 R(t,\tau) \left[\sum_m \delta(t - m\Delta) \sum_q \delta(\tau - 2q\Delta) + \right.$$

$$\left. + \sum_m \delta\left(t - (m + \tfrac{1}{2})\Delta\right) \sum_q \delta\left(\tau - (2q + 1)\Delta\right) \right] =$$

$$= \Delta^2 R(t,\tau) \sum_{\substack{n \quad \ell \\ n+\ell \text{ even}}} \delta(t - \tfrac{n\Delta}{2}) \delta(\tau - \ell\Delta) =$$

$$= \Delta^2 \sum_{\substack{n \quad \ell \\ n+\ell \text{ even}}} R(\tfrac{n\Delta}{2}, \ell\Delta) \delta(t - \tfrac{n\Delta}{2}) \delta(\tau - \ell\Delta) . \qquad (B-1)$$

That is, a couple of two-dimensional impulse trains, interspersed in the t,τ plane, have been applied, so as to use all the available information in (39). This is identical to the result for the TCF of impulsive time waveform $s_1(t)$, obtained by multiplying $s(t)$ by a sampling train; see appendix E.

We now define the corresponding WDF, CAF, and SCF as rigorous Fourier transforms of (B-1), using the standard forms, for all argument values,

$$W_1(t,f) = \int d\tau \, \exp(-i2\pi f\tau) \, R_1(t,\tau) \ ,$$

$$X_1(\nu,\tau) = \int dt \, \exp(-i2\pi\nu t) \, R_1(t,\tau) \ ,$$

$$A_1(\nu,f) = \int dt \, \exp(-i2\pi\nu t) \, W_1(t,f) =$$

$$= \int d\tau \, \exp(-i2\pi f\tau) \, X_1(\nu,\tau) =$$

$$= \iint dt \, d\tau \, \exp(-i2\pi\nu t - i2\pi f\tau) \, R_1(t,\tau) \ . \qquad (B-2)$$

These _exact_ interrelationships indicate that the _same_ SCF A_1 will result from TCF R_1, whether we proceed by way of the WDF or the CAF.

We have, in detail, WDF

$$W_1(t,f) = \int d\tau \, \exp(-i2\pi f\tau) \, R_1(t,\tau) =$$

$$= \frac{1}{2}\Delta \sum_m \delta(t - m\Delta) \int d\tau \, \exp(-i2\pi f\tau) \, R(t,\tau) \, 2\Delta \sum_q \delta(\tau - 2q\Delta) +$$

$$+ \frac{1}{2}\Delta \sum_m \delta\left(t - (m + \tfrac{1}{2})\Delta\right) \int d\tau \, \exp(-i2\pi f\tau) \, R(t,\tau) \, 2\Delta \sum_q \delta\left(\tau - (2q + 1)\Delta\right) =$$

$$= \frac{1}{2}\Delta \sum_m \delta(t - m\Delta) \, W(t,f) \overset{f}{\circledast} \sum_q \delta(f - \tfrac{q}{2\Delta}) +$$

$$+ \frac{1}{2}\Delta \sum_m \delta\left(t - (m + \tfrac{1}{2})\Delta\right) W(t,f) \overset{f}{\circledast} \sum_q (-1)^q \, \delta(f - \tfrac{q}{2\Delta}) =$$

$$= \frac{1}{2}\Delta \sum_m \delta(t - m\Delta) \sum_q W(t,f - \tfrac{q}{2\Delta}) +$$

$$+ \frac{1}{2}\Delta \sum_m \delta\left(t - (m + \tfrac{1}{2})\Delta\right) \sum_q (-1)^q \, W(t,f - \tfrac{q}{2\Delta}) = \qquad (B-3)$$

$$= \frac{1}{2} \Delta \sum_m \delta(t - m\Delta) \sum_q W(m\Delta, f - \frac{q}{2\Delta}) +$$

$$+ \frac{1}{2} \Delta \sum_m \delta\left(t - (m + \frac{1}{2})\Delta\right) \sum_q (-1)^q W\left((m + \frac{1}{2})\Delta, f - \frac{q}{2\Delta}\right) =$$

$$= \frac{1}{2} \Delta \sum_m \delta(t - m\Delta) W_a(m\Delta, f) + \frac{1}{2} \Delta \sum_m \delta\left(t - (m + \frac{1}{2})\Delta\right) W_a\left((m + \frac{1}{2})\Delta, f\right) =$$

$$= \frac{\Delta}{2} \sum_n \delta(t - \frac{n\Delta}{2}) W_a(\frac{n\Delta}{2}, f) , \qquad (B-4)$$

using (42) and (43). Thus, the areas of the impulses in W_1 are equal to the approximations W_a developed in (42) and (43), within a scale factor of $\Delta/2$.

Continuing on, from (B-3), the SCF is

$$A_1(v, f) = \int dt \exp(-12\pi v t) W_1(t, f) =$$

$$= \frac{1}{2} \sum_q \int dt \exp(-12\pi v t) W(t, f - \frac{q}{2\Delta}) \Delta \sum_m \delta(t - m\Delta) +$$

$$+ \frac{1}{2} \sum_q (-1)^q \int dt \exp(-12\pi v t) W(t, f - \frac{q}{2\Delta}) \Delta \sum_m \delta\left(t - (m + \frac{1}{2})\Delta\right) =$$

$$= \frac{1}{2} \sum_q A(\upsilon, f - \frac{q}{2\Delta}) \, \overset{\upsilon}{\circledast} \sum_m \delta(\upsilon - \frac{m}{\Delta}) \; +$$

$$+ \frac{1}{2} \sum_q (-1)^q A(\upsilon, f - \frac{q}{2\Delta}) \, \overset{\upsilon}{\circledast} \sum_m (-1)^m \delta(\upsilon - \frac{m}{\Delta}) \; =$$

$$= \frac{1}{2} \sum_q \sum_m A(\upsilon - \frac{m}{\Delta}, f - \frac{q}{2\Delta}) \; +$$

$$+ \frac{1}{2} \sum_q \sum_m (-1)^{q+m} A(\upsilon - \frac{m}{\Delta}, f - \frac{q}{2\Delta}) \; =$$

$$= \sum_{\substack{q \\ q+m}} \sum_{\substack{m \\ \text{even}}} A(\upsilon - \frac{m}{\Delta}, f - \frac{q}{2\Delta}) \qquad \text{for all } \upsilon, f \; . \qquad (B\text{-}5)$$

Thus, the SCF A_1, resulting from the impulsive sampling approach applied to the TCF, is not impulsive at all in the υ, f plane, and is identical with the approximations A_a developed in (51) and (62).

Proceeding instead via the CAF, we have

$$\chi_1(\upsilon, \tau) = \int dt \, \exp(-i2\pi\upsilon t) \, R_1(t, \tau) \; =$$

$$= \Delta \sum_q \delta(\tau - 2q\Delta) \int dt \, \exp(-i2\pi\upsilon t) \, R(t, \tau) \, \Delta \sum_m \delta(t - m\Delta) \; +$$

$$+ \Delta \sum_q \delta\left(\tau - (2q + 1)\Delta\right) \int dt \, \exp(-i2\pi\upsilon t) \, R(t, \tau) \, \Delta \sum_m \delta\left(t - (m + \tfrac{1}{2})\Delta\right) \; =$$

$$= \Delta \sum_q \delta(\tau - 2q\Delta) \; x(\nu,\tau) \overset{\nu}{\circledast} \sum_m \delta(\nu - \tfrac{m}{\Delta}) \; +$$

$$+ \Delta \sum_q \delta\left(\tau - (2q + 1)\Delta\right) \; x(\nu,\tau) \overset{\nu}{\circledast} \sum_m (-1)^m \; \delta(\nu - \tfrac{m}{\Delta}) \; =$$

$$= \Delta \sum_q \delta(\tau - 2q\Delta) \sum_m x(\nu - \tfrac{m}{\Delta}, \; \tau) \; +$$

$$+ \Delta \sum_q \delta\left(\tau - (2q + 1)\Delta\right) \sum_m (-1)^m \; x(\nu - \tfrac{m}{\Delta}, \; \tau) \; = \qquad (B\text{-}6)$$

$$= \Delta \sum_q \delta(\tau - 2q\Delta) \sum_m x\left(\nu - \tfrac{m}{\Delta}, \; 2q\Delta\right) \; +$$

$$+ \Delta \sum_q \delta\left(\tau - (2q + 1)\Delta\right) \sum_m (-1)^m \; x\left(\nu - \tfrac{m}{\Delta}, \; (2q + 1)\Delta\right) \; =$$

$$= \Delta \sum_q \delta(\tau - 2q\Delta) \; x_a(\nu, 2q\Delta) + \Delta \sum_q \delta\left(\tau - (2q + 1)\Delta\right) \; x_a\left(\nu, (2q + 1)\Delta\right) \; =$$

$$= \Delta \sum_n \delta(\tau - n\Delta) \; x_a(\nu, n\Delta) \; , \qquad (B\text{-}7)$$

via (56) and (57). Thus, the areas of the impulses in x_i are equal to the approximations x_a in (56) and (57), within a scale factor of Δ.

Now strictly speaking, there is no need to proceed to SCF A_1 via the CAF x_1, since (B-2) shows that there is only one A_1 function, regardless of how reached. Nevertheless, for completeness, we also present the last route. We have, using (B-6),

$$A_1(\upsilon,f) = \int d\tau \, \exp(-i2\pi f\tau) \, x_1(\upsilon,\tau) =$$

$$= \frac{1}{2} \sum_m \int d\tau \, \exp(-i2\pi f\tau) \, x(\upsilon - \frac{m}{\Delta},\tau) \, 2\Delta \sum_q \delta(\tau - 2q\Delta) +$$

$$+ \frac{1}{2} \sum_m (-1)^m \int d\tau \, \exp(-i2\pi f\tau) \, x(\upsilon - \frac{m}{\Delta},\tau) \, 2\Delta \sum_q \delta\big(\tau - (2q + 1)\Delta\big) =$$

$$= \frac{1}{2} \sum_m A(\upsilon - \frac{m}{\Delta},f) \overset{f}{\circledast} \sum_q \delta(f - \frac{q}{2\Delta}) +$$

$$+ \frac{1}{2} \sum_m (-1)^m A(\upsilon - \frac{m}{\Delta},f) \overset{f}{\circledast} \sum_q (-1)^q \delta(f - \frac{q}{2\Delta}) =$$

$$= \frac{1}{2} \sum_m \sum_q A(\upsilon - \frac{m}{\Delta}, f - \frac{q}{2\Delta}) + \frac{1}{2} \sum_m \sum_q (-1)^{m+q} A(\upsilon - \frac{m}{\Delta}, f - \frac{q}{2\Delta}) =$$

$$= \sum_{\substack{m \\ m+q}} \sum_{\substack{q \\ \text{even}}} A(\upsilon - \frac{m}{\Delta}, f - \frac{q}{2\Delta}) \quad \text{for all } \upsilon,f . \tag{B-8}$$

As anticipated, this is identical with (B-5). Thus we get a unique SCF in the υ,f plane.

APPENDIX C.   RECOVERY VIA DIRECT CONVOLUTION

This appendix is closely coupled with the previous one; it shows how to recover the original continuous two-dimensional TCF, CAF, and WDF from their impulsive counterparts. From (B-5), (B-8), (51), (62), (53), (54), and figure 6, the original SCF is

$$A(v,f) = A_1(v,f) \, D(v,f) \ . \qquad\qquad (C-1)$$

WDF RECOVERY

We have, using (B-4),

$$W(t,f) = \int dv \ \exp(i2\pi vt) \ A_1(v,f) \ D(v,f) =$$

$$= W_1(t,f) \overset{t}{\circledast} d(t,f) =$$

$$= \sum_n W_a(\tfrac{n\Delta}{2},f) \ \tfrac{\Delta}{2} \ d(t - \tfrac{n\Delta}{2},f) \qquad \text{for all } t,f \ , \qquad\qquad (C-2)$$

where

$$\tfrac{\Delta}{2} \ d(t,f) = \tfrac{\Delta}{2} \int dv \ \exp(i2\pi vt) \ D(v,f) =$$

$$= \left\{ \begin{array}{ll} \dfrac{\sin[2\pi \tfrac{t}{\Delta}(1 - 2\Delta|f|)]}{2\pi \tfrac{t}{\Delta}} & \text{for } |f| < \tfrac{1}{2\Delta} \\[4ex] 0 \quad \text{for } |f| > \tfrac{1}{2\Delta} \end{array} \right\} \quad \text{for all } t$$

$$= (1 - 2\Delta|f|) \ \text{sinc}\left[2 \tfrac{t}{\Delta}(1 - 2\Delta|f|)\right] \ \text{rect}(\Delta f) \ . \qquad\qquad (C-3)$$

These results agree with [6, (27) & (28)]. Interpolation rule (C-2) uses the available slices of information in the t,f plane of figure 8. A particular case of (C-3) is

$$\frac{\Delta}{2} d(0,f) = (1 - 2\Delta |f|) \, rect(\Delta f) . \qquad (C-4)$$

CAF RECOVERY

From (C-1) and (B-7), there follows

$$x(\nu,\tau) = \int df \, \exp(i2\pi f\tau) \, A_1(\nu,f) \, D(\nu,f) =$$

$$= x_1(\nu,\tau) \circledast \overset{\tau}{d}(\nu,\tau) =$$

$$= \sum_n x_a(\nu,n\Delta) \, \Delta \, d(\nu,\tau - n\Delta) \quad \text{for all } \nu,\tau , \qquad (C-5)$$

where

$$\Delta \, d(\nu,\tau) = \Delta \int df \, \exp(i2\pi f\tau) \, D(\nu,f) =$$

$$= \begin{cases} \dfrac{\sin[\pi \frac{\tau}{\Delta}(1 - \Delta |\nu|)]}{\pi \, \tau/\Delta} & \text{for } |\nu| < \frac{1}{\Delta} \\[4mm] 0 & \text{for } |\nu| > \frac{1}{\Delta} \end{cases} \qquad \text{for all } \tau$$

$$= (1 - \Delta |\nu|) \, sinc\left[\frac{\tau}{\Delta}(1 - \Delta |\nu|)\right] rect(\tfrac{1}{2} \Delta\nu) . \qquad (C-6)$$

Interpolation rule (C-5) uses the available slices of information in the ν,τ plane of figure 8. A special case of (C-6) is

$$\Delta \, d(\nu,0) = (1 - \Delta |\nu|) \, rect(\tfrac{1}{2} \Delta\nu) . \qquad (C-7)$$

TCF RECOVERY

From (C-1) and (B-1),

$$R(t,\tau) = \iint d\upsilon \, df \, \exp(i2\pi\upsilon t + i2\pi f\tau) \, A_1(\upsilon,f) \, D(\upsilon,f) =$$

$$= R_1(t,\tau) \circledast^{t\tau} \mathcal{D}(t,\tau) =$$

$$= \sum_n \sum_{\ell} R(\tfrac{n\Delta}{2}, \ell\Delta) \, \Delta^2 \mathcal{D}(t - \tfrac{n\Delta}{2}, \tau - \ell\Delta) \quad \text{for all } t,\tau , \qquad (C-8)$$
$$n+\ell \text{ even}$$

where

$$\Delta^2 \mathcal{D}(t,\tau) = \Delta^2 \iint d\upsilon \, df \, \exp(i2\pi\upsilon t + i2\pi f\tau) \, D(\upsilon,f) =$$

$$= \frac{\sin^2(\pi t/\Delta) - \sin^2(\tfrac{\pi\tau}{2\Delta})}{\tfrac{\pi^2}{\Delta^2}(t^2 - \tfrac{\tau^2}{4})} =$$

$$= \frac{\sin(\tfrac{\pi t}{\Delta}) + \sin(\tfrac{\pi\tau}{2\Delta})}{\tfrac{\pi}{\Delta}(t + \tfrac{\tau}{2})} \; \frac{\sin(\tfrac{\pi t}{\Delta}) - \sin(\tfrac{\pi\tau}{2\Delta})}{\tfrac{\pi}{\Delta}(t - \tfrac{\tau}{2})} \quad \text{for all } t,\tau . \qquad (C-9)$$

Particular values are

$$\Delta^2 \mathcal{D}(\pm \tfrac{\tau}{2}, \tau) = \text{sinc}(\tfrac{\tau}{\Delta}) . \qquad (C-10)$$

It should be observed that (C-8) dictates two-dimensional interpolation in the t,τ plane of figure 8. Attempts at simpler one-dimensional interpolation in t or τ alone are bound to fail.

PROPERTIES OF $\mathcal{D}(t,\tau)$

   The two-dimensional function $\mathcal{D}(t,\tau)$ is unlike any proposed previously for interpolation of TCF $R(t,\tau)$. Some of its properties are listed here.

$$\mathcal{D}(-t,\tau) = \mathcal{D}(t, -\tau) = \mathcal{D}(t,\tau) . \tag{C-11}$$

$$\Delta^2 \mathcal{D}(m\Delta, 2q\Delta) = \frac{\sin^2(\pi m) - \sin^2(\pi q)}{\pi^2(m^2 - q^2)} . \tag{C-12}$$

If $m \neq q$ or $m \neq -q$, then $\mathcal{D} = 0$. If $m = q$, then $t = \tau/2$, while if $m = -q$, then $t = -\tau/2$, giving

$$\Delta^2 \mathcal{D}(\pm q\Delta, 2q\Delta) = \text{sinc}(2q) . \tag{C-13}$$

Therefore $\mathcal{D}(m\Delta, 2q\Delta) = 0$ for all m,q, except that $\Delta^2 \mathcal{D}(0,0) = 1$. Similarly,

$$\mathcal{D}\left((m + \tfrac{1}{2})\Delta, (2q + 1)\Delta\right) = 0 \quad \text{for all } m,q . \tag{C-14}$$

   If we define

$$\mathcal{S}(t,\tau) = \mathcal{D}(t,2\tau) = \frac{\sin^2(\pi t/\Delta) - \sin^2(\pi \tau/\Delta)}{\pi^2(t^2 - \tau^2)} , \tag{C-15}$$

then

$$\mathcal{S}(\tau,t) = \mathcal{S}(t,\tau) , \tag{C-16}$$

meaning that $\mathcal{S}(t,\tau)$ is symmetric about the 45° line in the t,τ plane. Figure C-1 depicts some sample values of $\mathcal{D}$ in the first quadrant. In particular,

$$\Delta^2 \mathscr{D}\left(m\Delta, (2q + 1)\Delta\right) = \frac{-1}{\pi^2[m^2 - (q + \frac{1}{2})^2]} \ ,$$

$$\Delta^2 \mathscr{D}\left((m + \frac{1}{2})\Delta, 2q\Delta\right) = \frac{1}{\pi^2[(m + \frac{1}{2})^2 - q^2]} \ . \qquad (C-17)$$

Interpolation function $\mathscr{D}(t,\tau)$ decays <u>slowest</u> along the $t = \pm\tau/2$ lines in the t,τ plane, and <u>fastest</u> along the $t = 0$ and $\tau = 0$ lines. So direct interpolation of the TCF is not best approximated by horizontal or vertical slices, but in fact, by points between these slices.
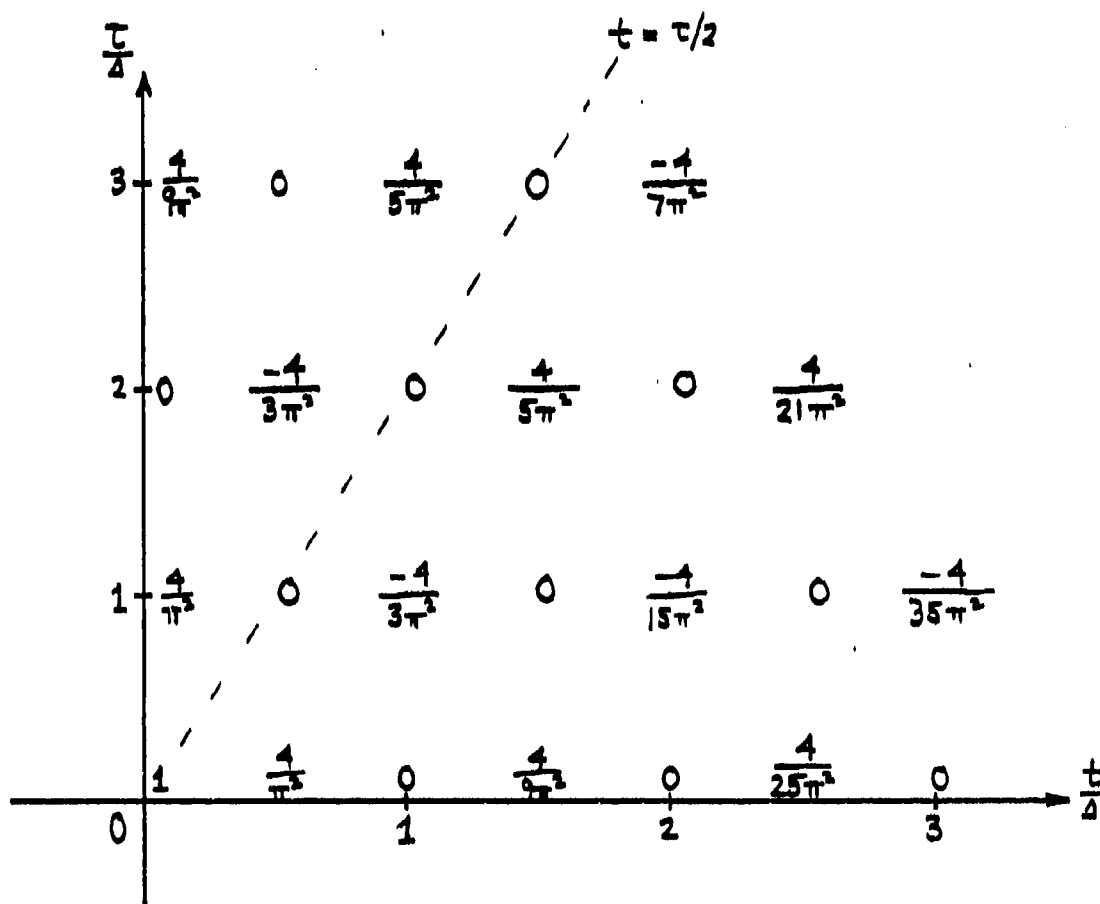

Figure C-1. Sample Values of $\Delta^2 \mathscr{D}(t,\tau)$

APPENDIX D. EVALUATION OF CAF

The original CAF of waveform s(t) is given by (73), in terms of spectrum $\tilde{S}(f)$ defined in (69). As in (87), we again presume that only the discrete frequency calculations

$$S(\tfrac{n}{N\Delta}) = \Delta \sum_k \exp(-i2\pi nk/N) \; s(k\Delta) \qquad \text{for } |n| < \tfrac{N}{2} \qquad (D-1)$$

are available. Since the frequency increment is $\Delta_f = (N\Delta)^{-1}$, we approximate CAF (73) according to

$$\tilde{x}(\nu,\tau) \equiv \frac{1}{N\Delta} \sum_n \exp(i2\pi \tfrac{n}{N\Delta} \tau) \; \tilde{S}(\tfrac{n}{N\Delta} + \tfrac{\nu}{2}) \; \tilde{S}^*(\tfrac{n}{N\Delta} - \tfrac{\nu}{2}) =$$

$$= \int df \; \exp(i2\pi f\tau) \; \tilde{S}(f + \tfrac{\nu}{2}) \; \tilde{S}^*(f - \tfrac{\nu}{2}) \frac{1}{N\Delta} \sum_n \delta(f - \tfrac{n}{N\Delta}) =$$

$$= x(\nu,\tau) \overset{\tau}{\circledast} \sum_n \delta(\tau - n\,N\Delta) =$$

$$= \sum_n x(\nu, \; \tau - n\,N\Delta) \qquad \text{for all } \nu,\tau . \qquad (D-2)$$

Since waveform s(t) is approximately limited to $|t| < T/2$ (see (84) and figure 9), then CAF $x(\nu,\tau)$ is approximately limited to $|\tau| < T$, as may be seen by substituting (32) into (34):

$$x(v,\tau) = \int dt \ \exp(-i2\pi vt) \ s(t + \tfrac{\tau}{2}) \ s^*(t - \tfrac{\tau}{2}) \ . \qquad (D\text{-}3)$$

Therefore, the approximate CAF $\tilde{x}$ takes the appearance shown in figure D-1. It is seen that overlap is negligible if we take

$$T < N\Delta - T, \quad \text{i.e.,} \quad N > \frac{2T}{\Delta} \ . \qquad (D\text{-}4)$$

This requirement on the FFT size in (D-1) is the same as that established in (93) for the approximate WDF $\tilde{W}$.

DISCRETIZATION IN v and τ

In order to utilize available samples (D-1) in the evaluation of approximation (D-2), we restrict the evaluation of the approximate CAF to frequency-shift values

$$\tilde{x}(\tfrac{2m}{N\Delta}, \ \tau) = \frac{1}{N\Delta} \sum_n \exp(i2\pi \tfrac{n}{N\Delta} \tau) \ \bar{s}(\tfrac{n+m}{N\Delta}) \ \bar{s}^*(\tfrac{n-m}{N\Delta}) \quad \text{for all } \tau \ . \qquad (D\text{-}5)$$

Furthermore, we consider only the particular values of time delay given by

$$\tilde{x}(\tfrac{2m}{N\Delta}, \ q\Delta) = \frac{1}{N\Delta} \sum_n \exp(i2\pi nq/N) \ S(\tfrac{n+m}{N\Delta}) \ S^*(\tfrac{n-m}{N\Delta}) \ . \qquad (D\text{-}6)$$

since the right-hand side is now an N-point FFT for each m value of interest. N values of q are swept out by each FFT. Compare (D-6) with (111).
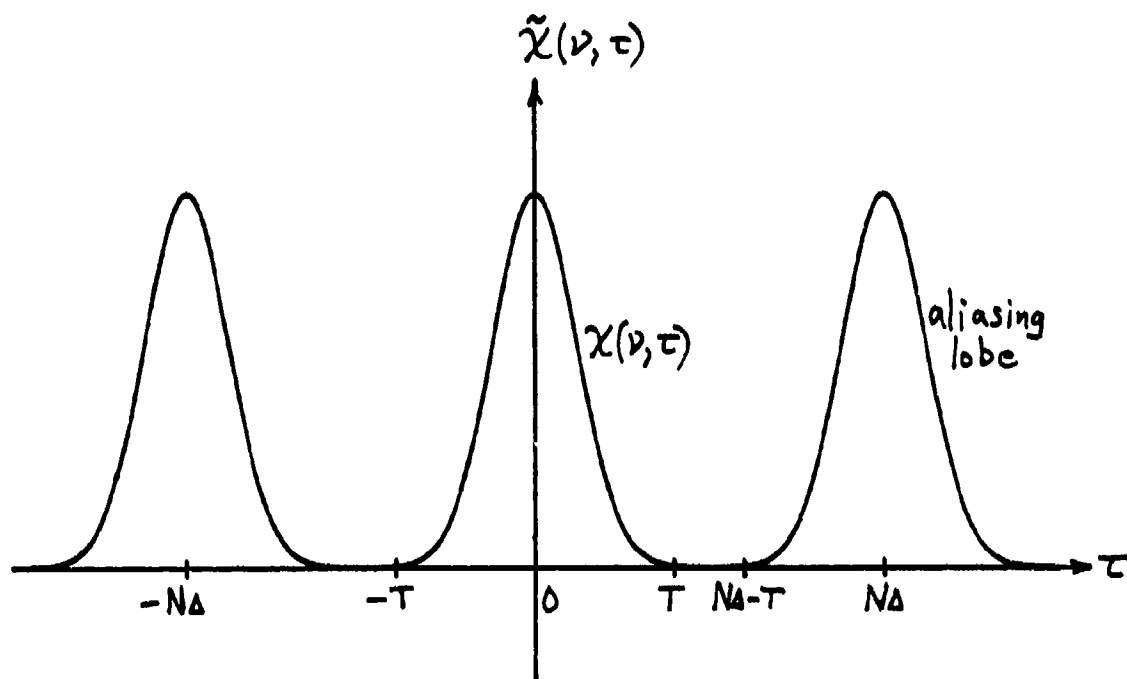
Figure D-1. Delay-Aliased CAF

APPENDIX E. TCF OF IMPULSIVELY-SAMPLED WAVEFORM

Suppose continuous waveform $s(t)$ is sampled with an infinite impulse train (with delay t_0) yielding impulsive waveform

$$s_1(t) = s(t) \, \Delta \sum_k \delta(t - t_0 - k\Delta) . \tag{E-1}$$

The corresponding TCF is

$$R_1(t,\tau) \equiv s_1(t + \tfrac{\tau}{2}) \, s_1^*(t - \tfrac{\tau}{2}) =$$

$$= R(t,\tau) \, \Delta^2 \sum_k \sum_m \delta(t - t_0 + \tfrac{\tau}{2} - k\Delta) \, \delta(t - t_0 - \tfrac{\tau}{2} - m\Delta) . \tag{E-2}$$

This function has impulses in the t,τ plane at

$$\left. \begin{aligned} t - t_0 + \tfrac{\tau}{2} &= k\Delta \\[2mm] t - t_0 - \tfrac{\tau}{2} &= m\Delta \end{aligned} \right\} , \quad \text{i.e., at} \quad \begin{cases} t = t_0 + \dfrac{k+m}{2}\,\Delta \\[3mm] \tau = (k - m)\Delta . \end{cases} \tag{E-3}$$

Furthermore, the area of each of these impulses is 1:

$$\iint dt \, d\tau \, \delta(t - t_0 + \tfrac{\tau}{2} - k\Delta) \, \delta(t - t_0 - \tfrac{\tau}{2} - m\Delta) = \int d\tau \, \delta(\tau + m\Delta - k\Delta) = 1 . \tag{E-4}$$

So (E-2) can be expressed alternatively as

$$R_1(t,\tau) = R(t,\tau) \, \Delta^2 \sum_k \sum_m \delta(t - t_0 - \tfrac{k+m}{2}\,\Delta) \, \delta(\tau - (k - m)\Delta) . \tag{E-5}$$

Now let

$$n = k + m , \quad \ell = k - m . \tag{E-6}$$

Then $n \pm \ell$ must be even, giving

$$R_1(t,\tau) = R(t,\tau) \, \Delta^2 \sum_{\substack{n \\ n+\ell \text{ even}}} \sum_{\ell} \delta(t - t_0 - \frac{n}{2} \Delta) \, \delta(\tau - \ell\Delta) =$$

$$= \Delta^2 \sum_{\substack{n \\ n+\ell \text{ even}}} \sum_{\ell} R(t_0 + \frac{n}{2} \Delta, \, \ell\Delta) \, \delta(t - t_0 - \frac{n}{2} \Delta) \, \delta(\tau - \ell\Delta) . \tag{E-7}$$

This is a slight generalization of (B-1), to allow for delayed sampling. Thus, the two approaches, (B-1) and (E-1), yield identical results.

FUNDAMENTAL TWO-DIMENSIONAL SAMPLING PATTERNS

Suppose, in (B-1), that we let

$$R(t,\tau) = 1 \quad \text{for all } t, \tau . \tag{E-8}$$

Then $R_1(t,\tau)$ there approaches

$$r_1(t,\tau) \equiv \Delta^2 \sum_{\substack{n \\ n+\ell \text{ even}}} \sum_{\ell} \delta(t - \frac{n\Delta}{2}) \, \delta(\tau - \ell\Delta) . \tag{E-9}$$

But, at the same time, use of (E-8) in SCF (35) yields

$$A(\nu,f) = \delta(\nu) \, \delta(f) , \tag{E-10}$$

while (B-5) and (B-8) approach

$$a_1(v,f) = \sum_{\substack{q \\ q+m \ \text{even}}} \sum_{m} \delta\left(v - \frac{m}{\Delta}\right) \delta\left(f - \frac{q}{2\Delta}\right) .$$ (E-11)

Thus, (E-9) and (E-11) are a double Fourier transform pair:

$$a_1(v,f) = \iint dt \ d\tau \ \exp(-12\pi vt - 12\pi f\tau) \ r_1(t,\tau) .$$ (E-12)

They generalize one-dimensional result (7) & (8) to two dimensions with interspersed sampling. The impulse patterns of $r_1(t,\tau)$ and $a_1(v,f)$ in their respective domains are displayed in figures E-1 and E-2.
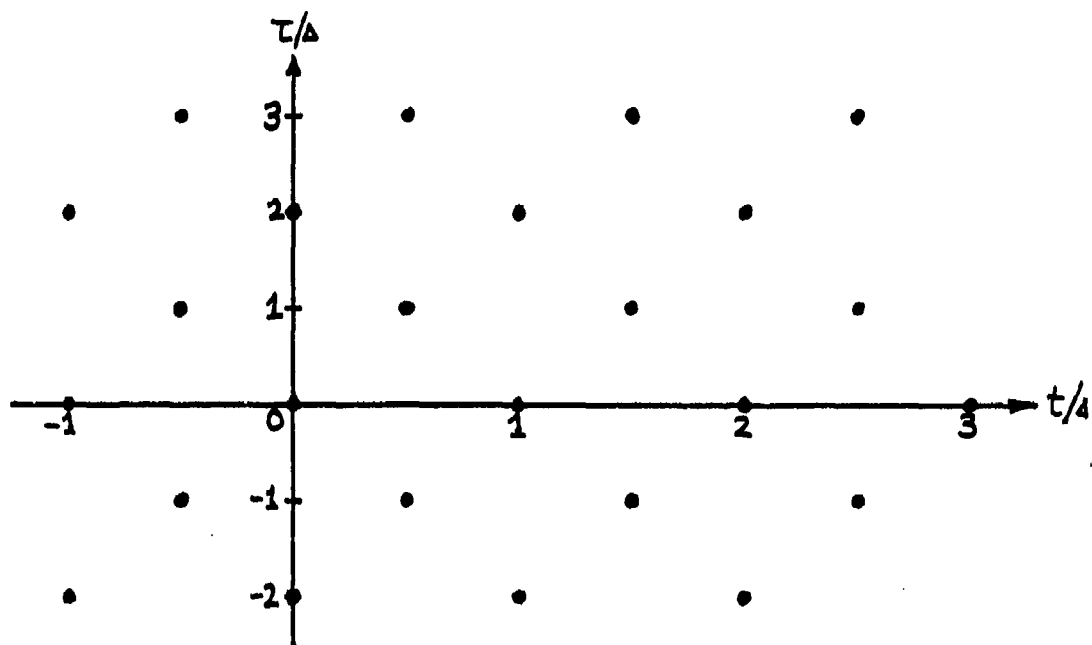
Figure E-1.   Impulse Locations for $r_1(t,\tau)$ in (E-9)
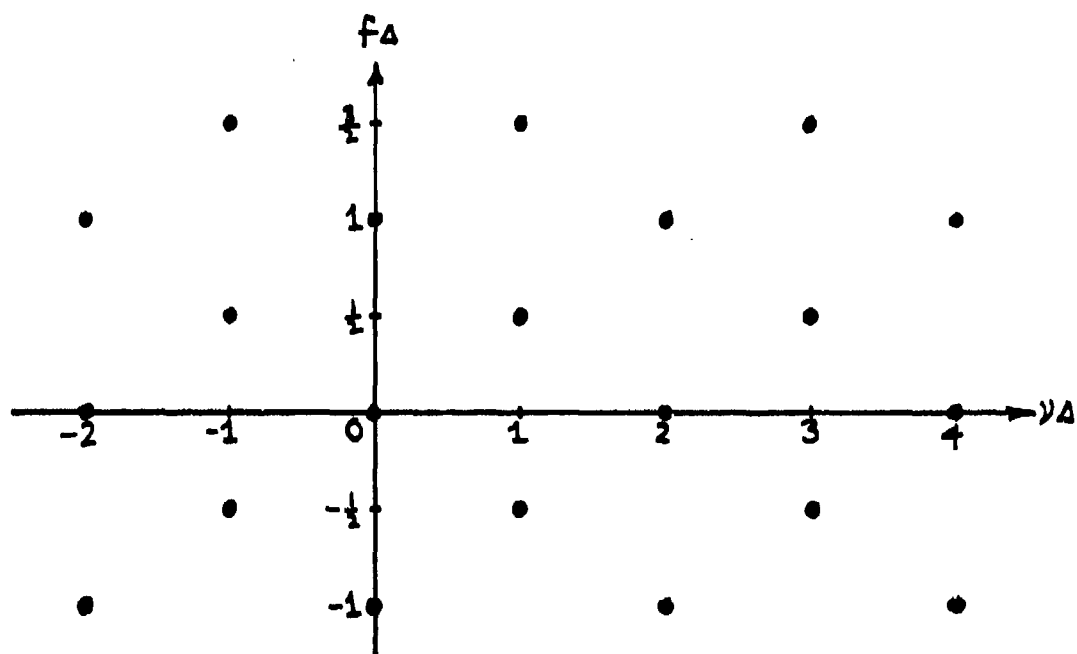
Figure E-2.   Impulse Locations for $a_1(v,f)$ in (E-11).

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